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ERRATA.

p. 88, line 13 from top, read $\bar{\omega}_1$ instead of ω_1.
p. 93, " 19 " " $\bar{z}_2$ " " $\bar{z}$.
p. 94, " 17 " " as " " in.
p. 99, " 11 " " $\log \omega_i'$ instead of $\log \omega i$.
p. 99, " 11 " " μ_i " " μi.
p. 99, last line, contiguas " " contigues.
p. 100, line 9 " " $x^{1-\gamma}$ " " $\alpha^{1-\gamma}$.
p. 101, foot-note, Schlesinger instead of Schleisinger.
p. 102, line 12, " " coefficients " " functions.
p. 103, foot-note, $d^{(i_\lambda)}y$ instead of $d^{(i_k)}y$ and $y^{(i_\lambda)}$ instead of $y^{(i_\lambda}$.
p. 106, line 2 from top, the word *if* is omitted at the beginning of the line.

ADDENDUM.

p. 97, line 24, after the word *Then* add the words, *$y_1, \ldots, y_n$ undergo the substitution.*

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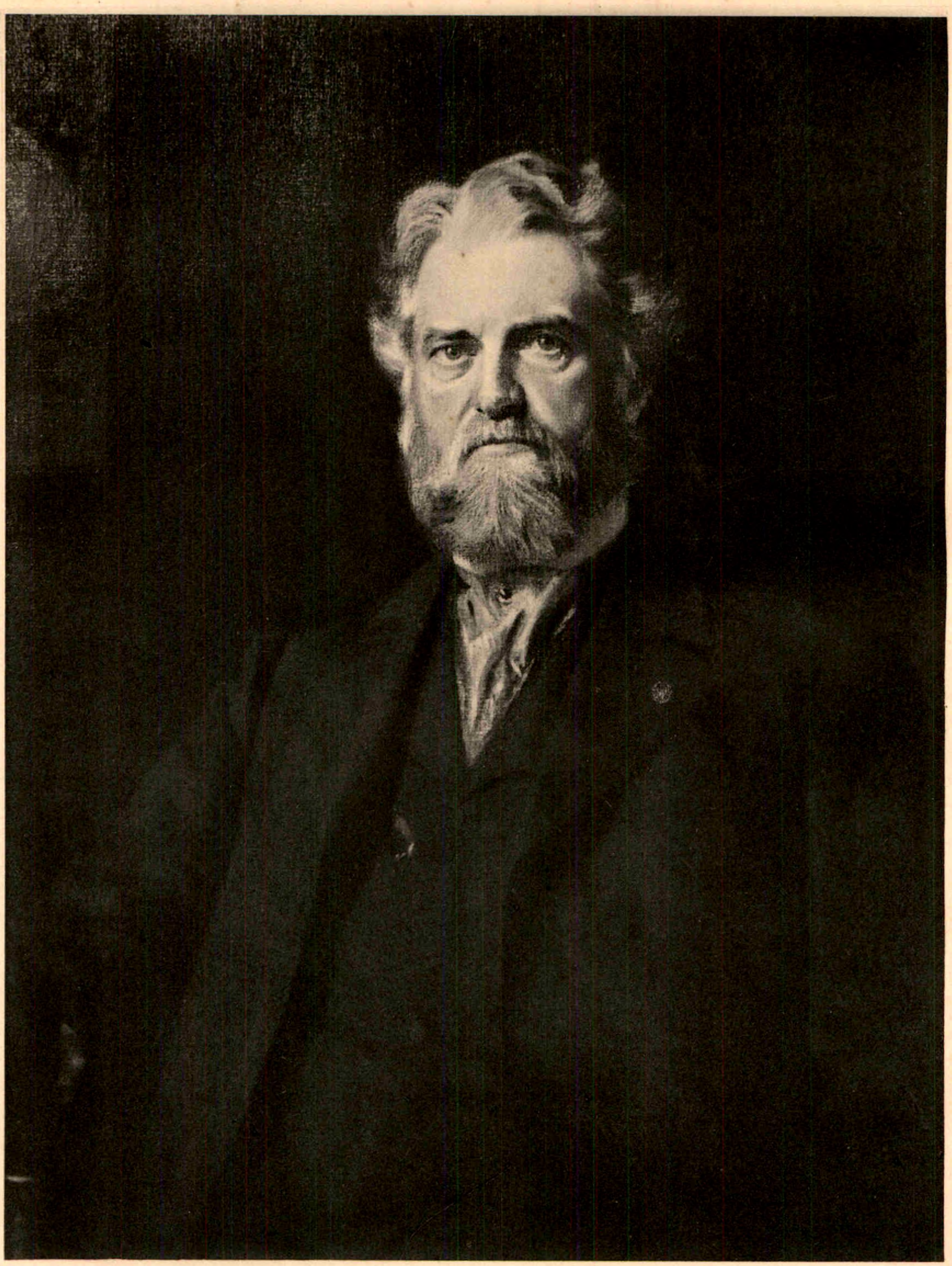

S. Newcomb

Systems of Revolution and their Relation to Conical Systems, in the Theory of Lamé's Products.

BY F. H. SAFFORD.

SECTION I.

*Introduction.**

The problem treated in the first part of the present paper is an application of a theorem due to Lord Kelvin, by means of which, from a known solution, V, of Laplace's Equation in terms of coördinates corresponding to a system of mutually orthogonal surfaces, a solution may be readily deduced for a new system of surfaces obtained from the first by inversion.

This theorem has been used in an extended sense, so that real surfaces have been obtained from imaginary surfaces by inversions with regard to imaginary points as centres.

Definition of Lamé's Products.

In problems requiring the solution of Laplace's Equation, it is often possible to obtain a solution by transferring to curvilinear coördinates, λ, μ, ν, and assuming that V is a product of three factors, i. e.

$$V = L . M . N, \tag{1}$$

where L, M, N are functions of λ, μ, ν respectively. Such an expression for V is called a Lamé's Product.

In cases where a solution in the form of (1) cannot be obtained, it is sometimes possible to obtain a solution on the assumption that

$$V = T . L . M . N, \tag{2}$$

where L, M, N have the same meaning as before. But T is a factor which is a function of more than one of the quantities λ, μ, ν, and, moreover, is independent of the accessory parameters which occur in L, M, N. These parameters first appear when the new form of Laplace's Equation is resolved into ordinary differential equations.

---

*The writer gratefully acknowledges his indebtedness to Professor Maxime Bôcher, of Harvard University, for suggestions and criticisms.

Several topics in the first section of this paper have been discussed by A. Wangerin with the aid of elliptic functions. In the following pages the use of elliptic functions has been avoided, while a simpler and more symmetrical notation than is ordinarily used by writers on this subject has been adopted. Wangerin's earliest paper* gives the ordinary differential equations obtained when the extraneous factor is $\frac{1}{\sqrt{r}}$. Another paper by the same author will be mentioned later.

Theorem of Lord Kelvin.

The theorem of Lord Kelvin, before referred to, shows that if $V(xyz)$ is a solution of Laplace's Equation, then

$$\frac{1}{\sqrt{x^2+y^2+z^2}}\cdot V\left[\frac{\rho x}{x^2+y^2+z^2},\ \frac{\rho y}{x^2+y^2+z^2},\ \frac{\rho z}{x^2+y^2+z^2}\right] \tag{3}$$

is also a solution. Here space has been subjected to an inversion with regard to a sphere of radius $\sqrt{\rho}$ described about the origin.

If then there are given three families of mutually orthogonal surfaces,

$$F_1(xyz)=\lambda, \tag{4}$$
$$F_2(xyz)=\mu, \tag{5}$$
$$F_3(xyz)=\nu, \tag{6}$$

and if in terms of these curvilinear coördinates λ, μ, ν a solution of Laplace's Equation be obtained in the form of a Lamé's Product, then, by applying Lord Kelvin's theorem, Lamé's Products (with an extraneous factor) may be at once obtained for the system of surfaces derived from (4)(5)(6) by inversion.

Surfaces of Revolution obtained from Cones of the Second Degree.

The starting point here is a system of confocal cones of the second degree and concentric spheres.

---

* Reduction der Potentialgleichung. Preisschriften der Fürstlich Jablnowskischen Gesellschaft der Wissenschaften. No. 18. Leipzig, 1875.

This primary system of surfaces will be transformed by inversion into a system of revolution.

Then since Lamé's products

$$V = L . M . N \tag{7}$$

can be found for the cone system, solutions of Laplace's Equation may be deduced from them for the new system in the form

$$V = T . L . M . N, \tag{8}$$

where T is an extraneous factor briefly treated above. The first topic is the transformation of the cone system into a system of revolution. The equations of the cone system may be written

$$x^2 + y^2 + z^2 = \lambda, \tag{9}$$

$$\frac{x^2}{\mu - \varepsilon_1} + \frac{y^2}{\mu - \varepsilon_2} + \frac{z^2}{\mu - \varepsilon_3} = 0, \tag{10}$$

$$\frac{x^2}{\nu - \varepsilon_1} + \frac{y^2}{\nu - \varepsilon_2} + \frac{z^2}{\nu - \varepsilon_3} = 0. \tag{11}$$

In order to obtain from this system a system of revolution, it is necessary and sufficient that the concentric spheres shall be transformed into meridian planes.

An inversion of the family of concentric spheres with regard to any point except their centre will give a family of eccentric spheres, having a common circle of intersection, real or imaginary. A second inversion with regard to a point on the circle of intersection will transform this circle into a right line. This right line is, accordingly, the axis of the planes into which the eccentric spheres have been transformed.

The explicit formulæ of transformation will now be obtained by means of a detailed transformation of the concentric spheres. It will simplify the formulæ for an inversion if the origin be first changed to the chosen centre of inversion.

The centre of the first inversion will be taken at any point in space (l, m, n) where all three coördinates are not zero.

The formulæ of transformation are

$$x = x_1 + l, \quad y = y_1 + m, \quad z = z_1 + n. \tag{12}$$

The family of concentric spheres (9) is now

$$(x_1 + l)^2 + (y_1 + m)^2 + (z_1 + n)^2 = \lambda. \tag{13}$$

The next transformation is an inversion with regard to a unit sphere whose centre is at the origin, i. e.

$$x_1 = \frac{x_2}{x_2^2 + y_2^2 + z_2^2} \quad y_1 = \frac{y_2}{x_2^2 + y_2^2 + z_2^2} \quad z_1 = \frac{z_2}{x_2^2 + y_2^2 + z_2^2}. \tag{14}$$

The result is the following family of eccentric spheres:

$$(x_2^2 + y_2^2 + z_2^2)(l^2 + m^2 + n^2 - \lambda) + 2lx_2 + 2my_2 + 2nz_2 + 1 = 0. \tag{15}$$

The centre for the second inversion is the point (a, b, c), which is to be taken on the circle of intersection of the eccentric spheres.

It follows that a, b, c satisfy (15) for all values of λ, giving

$$a^2 + b^2 + c^2 = 0, \tag{16}$$

$$2al + 2bm + 2cn + 1 = 0. \tag{17}$$

The equations for the change of origin are

$$x_2 = x_3 + a \quad y_2 = y_3 + b \quad z_2 = z_3 + c. \tag{18}$$

By the use of (16)(17)(18) the equation of the eccentric spheres (15) becomes

$$(x_3^2 + y_3^2 + z_3^2 + 2ax_3 + 2by_3 + 2cz_3)(l^2 + m^2 + n^2 - \lambda) + 2lx_3 + 2my_3 + 2nz_3 = 0. \tag{19}$$

The second inversion will be made with regard to a sphere whose radius is $\sqrt{\rho}$, i. e.

$$x_3 = \frac{\rho x_4}{x_4^2 + y_4^2 + z_4^2} \quad y_3 = \frac{\rho y_4}{x_4^2 + y_4^2 + z_4^2} \quad z_3 = \frac{\rho z_4}{x_4^2 + y_4^2 + z_4^2}. \tag{20}$$

Equation (19) becomes

$$(2ax_4 + 2by_4 + 2cz_4 + \rho)(l^2 + m^2 + n^2 - \lambda) + 2(lx_4 + my_4 + nz_4) = 0. \tag{21}$$

Equation (21) represents a family of planes whose axis is the intersection of the following planes:

$$2ax_4 + 2by_4 + 2cz_4 + \rho = 0, \tag{22}$$

$$lx_4 + my_4 + nz_4 = 0. \tag{23}$$

It will be simpler to choose the coefficients in (22) and (23), so that the axis of the family of planes shall be perpendicular to the $y_4 z_4$ plane. The condition equation is

$$l = a = 0. \tag{24}$$

The origin is next transferred to a point (d, e, f) taken on the axis at its intersection with the $y_4 z_4$ plane. Thence come the condition equations

$$2ad + 2be + 2cf + \rho = 0, \tag{25}$$

$$ld + me + nf = 0, \tag{26}$$

$$d = 0. \tag{27}$$

The equations for the change of origin are

$$x_4 = x_5 + d \quad y_4 = y_5 + e \quad z_4 = z_5 + f. \tag{28}$$

By the use of (24) . . . (28) the family of planes, (21), takes the final form, omitting subscripts,

$$y - z \tan\phi = 0, \tag{29}$$

$$\tan\phi = \frac{c\,(1 - 4b^2\lambda)}{b\,(1 + 4b^2\lambda)}. \tag{30}$$

It will simplify several of the later equations and involve no loss in generality to assume the equation

$$4b^2 - 1 = 0. \tag{31}$$

The transformation and condition equations may now be reduced to the form

$$4b^2 = -4c^2 = 1, \tag{32}$$

$$x = \frac{x_5}{2\,(by_5 + cz_5)}, \tag{33}$$

$$y = \frac{b}{2\rho\,(by_5 + cz_5)}\,(x_5^2 + y_5^2 + z_5^2 - \rho^2), \tag{34}$$

$$z = \frac{c}{2\rho\,(by_5 + cz_5)}\,(x_5^2 + y_5^2 + z_5^2 + \rho^2), \tag{35}$$

$$x^2 + y^2 + z^2 = \frac{-(by_5 - cz_5)}{(by_5 + cz_5)}. \tag{36}$$

The inverse of the above transformations will be of use later and is

$$x^5 = \frac{-\rho x}{2\,(by + cz)}, \tag{37}$$

$$y_5 = \frac{\rho b}{2\,(by + cz)}\,(x^2 + y^2 + z^2 - 1), \tag{38}$$

$$z_5 = \frac{\rho c}{2\,(by + cz)}\,(x^2 + y^2 + z^2 + 1), \tag{39}$$

$$x_5^2 + y_5^2 + z_5^2 = -\frac{\rho^2\,(by - cz)}{(by + cz)}. \tag{40}$$

The transformations defined by (32) . . . (36) have been shown to change the concentric spheres of (9) into meridian planes. Applying the same transformations to the cones of (10) and (11) gives, omitting subscripts,

$$\frac{4\rho^2x^2}{\mu-\varepsilon_1}+\frac{(x^2+y^2+z^2-\rho^2)^2}{\mu-\varepsilon_2}-\frac{(x^2+y^2+z^2+\rho^2)^2}{\mu-\varepsilon_3}=0, \tag{41}$$

and a similar equation with μ replaced by ν.

Equations (29) and (41) define a system of revolution obtained from the primary system of cones and spheres.

Rotation Cyclids.

Equation (41) defines a class of surfaces known as cyclids of rotation,* and may be thrown into a more familiar form by a linear transformation of μ, ε_1, ε_2, ε_3 defined by the equations

$$\mu=\frac{\alpha\bar{\mu}+\beta}{\gamma\bar{\mu}+\delta}\quad \gamma\bar{\varepsilon}_2+\delta=0, \tag{42}$$

$$\varepsilon_1=\frac{\alpha\bar{\varepsilon}_1+\beta}{\gamma\bar{\varepsilon}_1+\delta}. \tag{43}$$

By transformation of the same type as (43), ε_2 and ε_3 are replaced by $\bar{\varepsilon}_3$ and $\bar{\varepsilon}_4$ respectively.

The result is

$$\frac{4\rho^2x^2}{\bar{\mu}-\bar{\varepsilon}_1}+\frac{4\rho^2(y^2+z^2)}{\bar{\mu}-\bar{\varepsilon}_2}+\frac{(x^2+y^2+z^2-\rho^2)^2}{\bar{\mu}-\bar{\varepsilon}_3}-\frac{(x^2+y^2+z^2+\rho^2)^2}{\bar{\mu}-\bar{\varepsilon}_4}=0. \tag{44}$$

In a later section it will be necessary to have the values of x^2 and $r^2=y^2+z^2$ obtained from (44) and the similar equation where ν replaces μ. The direct solution is possible but extremely long, so that it is more convenient to proceed in the following manner.

---

* The possibility of obtaining cyclids of rotation in the manner shown appears from the table on p. 65 in Professor Böcher's book "Ueber die Reihenentwickelungen der Potentialtheorie," Taubner, Leipzig, 1894.

From (9)(10)(11),

$$x^2 = \frac{\lambda\,(\mu - \varepsilon_1)(\nu - \varepsilon_1)}{(\varepsilon_2 - \varepsilon_1)(\varepsilon_3 - \varepsilon_1)}, \tag{45}$$

$$y^2 = \frac{\lambda\,(\mu - \varepsilon_2)(\nu - \varepsilon_2)}{(\varepsilon_3 - \varepsilon_2)(\varepsilon_1 - \varepsilon_2)}, \tag{46}$$

$$z^2 = \frac{\lambda\,(\mu - \varepsilon_3)(\nu - \varepsilon_3)}{(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)}. \tag{47}$$

From (32)(37) . . . (40),

$$x_5^2 = \frac{\rho^2 x^2}{(y \pm zi)^2}, \tag{48}$$

$$r_5^2 = y_5^2 + z_5^2 = -\frac{\rho^2\,(x^2 + y^2 + z^2)}{(y \pm zi)^2}. \tag{49}$$

Since the coördinate axes of the cyclid system are the $x_5 y_5 z_5$ axes, it is necessary merely to express x_5^2 and r_5^2 in terms of μ and ν by means of (42)(43)(45)(46)(47). The result is, dropping dashes and subscripts,

$$x^2 = \frac{\rho^2\,(\mu - \varepsilon_1)(\nu - \varepsilon_1)(\varepsilon_2 - \varepsilon_3)(\varepsilon_2 - \varepsilon_4)(\varepsilon_3 - \varepsilon_4)}{(\varepsilon_1 - \varepsilon_2)\left[\sqrt{(\mu - \varepsilon_3)(\nu - \varepsilon_3)(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_4)} \pm \sqrt{(\mu - \varepsilon_4)(\nu - \varepsilon_4)(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)}\right]^2}, \tag{50}$$

$$r^2 = \frac{\rho^2\,(\mu - \varepsilon_2)(\nu - \varepsilon_2)(\varepsilon_1 - \varepsilon_3)(\varepsilon_1 - \varepsilon_4)(\varepsilon_3 - \varepsilon_4)}{(\varepsilon_2 - \varepsilon_1)\left[\sqrt{(\mu - \varepsilon_3)(\nu - \varepsilon_3)(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_4)} \pm \sqrt{(\mu - \varepsilon_4)(\nu - \varepsilon_4)(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)}\right]^2}. \tag{51}$$

*Laplace's Equation in conical coördinates.**

It is now desired to obtain V of Laplace's Equation in the form of Lamé's Products and an extraneous factor, the parameters in V corresponding to the surfaces of the system of revolution (29)(41). For this purpose it is necessary to obtain V first for the cone system in the form of Lamé's Products.

From (45)(46)(47)

$$ds^2 = \tfrac{1}{4}\left[\frac{d\lambda^2}{\lambda} + \frac{\lambda\,(\nu - \mu)\,d\mu^2}{(\mu - \varepsilon_1)(\mu - \varepsilon_2)(\mu - \varepsilon_3)} + \frac{\lambda\,(\mu - \nu)\,d\nu^2}{(\nu - \varepsilon_1)(\nu - \varepsilon_2)(\nu - \varepsilon_3)}\right]. \tag{52}$$

---

* The results stated in this section are familiar, although usually given in a somewhat different notation. See Lamé's original memoir in Liouville's Journal, vol. 4, p. 187.

In place of λ, μ, ν are to be introduced u, v, w, defined as follows:

$$u = \int \frac{d\lambda}{2\sqrt{\lambda^3}}, \tag{53}$$

$$v = \int \frac{d\mu}{2\sqrt{(\mu - \varepsilon_1)(\mu - \varepsilon_2)(\mu - \varepsilon_3)}}, \tag{54}$$

$$w = \int \frac{d\nu}{2\sqrt{(\nu - \varepsilon_1)(\nu - \varepsilon_2)(\nu - \varepsilon_3)}}. \tag{55}$$

Equation (52) becomes

$$ds^2 = \lambda^2 du^2 + \lambda(\nu - \mu) dv^2 + \lambda(\mu - \nu) dw^2. \tag{56}$$

Laplace's Equation in curvilinear coördinates u, v, w is

$$D_u\left(\frac{h_1}{h_2 h_3} D_u V\right) + D_v\left(\frac{h_2}{h_3 h_1} D_v V\right) + D_w\left(\frac{h_3}{h_1 h_2} D_w V\right) = 0. \tag{57}$$

The values of h_1^2, h_2^2, h_3^2 are the reciprocals of the coefficients of du^2, dv^2, dw^2 respectively in (56).

Equation (57) is now written

$$\frac{1}{\lambda} D_u^2 V + \frac{1}{\nu - \mu} D_v^2 V + \frac{1}{\mu - \nu} D_w^2 V = 0. \tag{58}$$

Letting

$$V = L.M.N, \tag{59}$$

(58) is resolved by the usual method into the three equations

$$\frac{d^2 L}{du^2} - (A\lambda) L = 0, \tag{60}$$

$$\frac{d^2 M}{dv^2} - (A\mu + B) M = 0, \tag{61}$$

$$\frac{d^2 N}{dw^2} - (A\nu + B) N = 0. \tag{62}$$

The general solution of (60) is

$$L = A_1 \lambda^{\frac{m}{2}} + B_1 \lambda^{-\frac{m+1}{2}}, \tag{63}$$

where m is thus defined:

$$A = m(m+1). \tag{64}$$

Equations (61) and (62) are Lamé's Equations, whose solutions may be indicated thus:

$$M = E_1(\mu), \quad N = E_2(\nu). \tag{65}$$

It follows that (58) has solutions in the form of Lamé's Products, i. e.

$$V = L.M.N. = (A_1\lambda^{\frac{m}{2}} + B_1\lambda^{-\frac{m+1}{2}})\,E_1(\mu)\,E_2(\nu). \tag{66}$$

Application of the Theorem of Lord Kelvin.

Since (66) gives a value of V in terms of parameters corresponding to the surfaces of the cone system, the value of V for the system of rotation, (29)(41) is written at once by the theorem of Lord Kelvin,

$$V = T.(A_1\lambda^{\frac{m}{2}} + B_1\lambda^{-\frac{m+1}{2}})\,E_1(\mu)\,E_2(\nu). \tag{67}$$

The parameters λ, μ, ν correspond now to the surfaces of the system of rotation.

It remains to calculate the factor T, and this is most easily done by the aid of the transformation formulæ in the detailed form given by (12)(14)(16)(17)(18) (20)(24) . . . (28)(31).

Starting with (59) the first inversion gives

$$V = \frac{1}{\sqrt{x_2^2 + y_2^2 + z_2^2}}\,.\,L.M.N, \tag{68}$$

while the result of the other changes is

$$V = \frac{1}{\sqrt{2\rho\,(by_5 + cz_5)}}\,.\,L.M.N. \tag{69}$$

Equation (69) may be improved in form by replacing the parameter λ by the angle between any meridian plane and the plane $x_5 z_5$. This angle has been denoted by ϕ, and from (30)(31)

$$\lambda = (\cos\phi \pm i\sin\phi)^2. \tag{70}$$

Also let R be the distance from the axis of rotation to any point $(x_5 y_5 z_5)$, then it follows that

$$y_5 = R\sin\phi, \quad z_5 = R\cos\phi, \tag{71}$$

$$\frac{1}{\sqrt{2\rho\,(by_5 + cz_5)}} = \frac{1}{\sqrt{R}}\cdot\frac{1}{\sqrt{2\rho\,(b\sin\phi + c\cos\phi)}}$$

$$= \frac{c_0}{\sqrt{R}}(\cos\phi \pm i\sin\phi)^{\frac{1}{2}}. \tag{72}$$

Making in (69) the substitutions indicated in (70)(72) and giving to L, M, N the values in (66), the result is

$$V = \frac{1}{\sqrt{R}} \left[A_0 \cos\left(m+\tfrac{1}{2}\right)\phi + B_0 \sin\left(m+\tfrac{1}{2}\right)\phi \right] E_1(\mu)\, E_2(\nu). \tag{73}$$

Equation (73) gives the desired value of V in the form of Lamé's Products and an extraneous factor, the parameters ϕ, μ, ν corresponding to the surfaces of the system of revolution derived from the cone system, as previously explained. The extraneous factor involves R only, i. e. the distance of any point from the axis of revolution.

The accessory parameters in $E_1(\mu)$ and $E_2(\nu)$ are A, which equals $m(m+1)$, and B.

The most general system of revolution obtainable from the given cone system.

Returning to the system of revolution, (29)(41), it is evident that any further transformation which gives a rotation system must leave the axis unchanged. It is then expressible as an inversion with regard to a point on the axis, accompanied by a translation along the axis, the equations being

$$x = \frac{a(\bar{x}+q)}{(\bar{x}+q)^2+\bar{y}^2+\bar{z}} + p, \quad y = \frac{a\bar{y}}{(\bar{x}+q)^2+\bar{y}^2+\bar{z}^2}\, z = \frac{a\bar{z}}{(\bar{x}+q)^2+\bar{y}^2+\bar{z}^2}. \tag{74}$$

The resulting system of revolution is the most general type obtainable from the cones of the second degree defined by (10) and (11).

The same result may be reached in a different way by letting the axis of revolution make the angles α_1, β_1, γ_1, with the x, y, z axes in (9)(10)(11), and then making the transformations defined by (32) . . . . (36).

The following relations will be useful for reference:

$$\cos\alpha_1 = \frac{2pq+a}{a}, \tag{75}$$

$$\cos\beta_1 = \frac{p^2q+pa-\rho^2q}{\rho a}, \tag{76}$$

$$\cos\gamma_1 = \frac{\pm i(p^2q+pa+\rho^2q)}{\rho a}, \tag{77}$$

and conversely

$$p = \frac{\rho(1 - \cos\alpha_1)}{\cos\beta_1 \pm i\cos\gamma_1}, \tag{78}$$

$$q = -\frac{a}{2\rho}(\cos\beta_1 \pm i\cos\gamma_1). \tag{79}$$

SECTION II.

Determination of the most general surfaces of revolution generated by confocal cyclic curves, for which Lamé's Products, with the extraneous factor $\frac{1}{\sqrt{r}}$*, exist.*

In this section will be considered surfaces of revolution whose meridian curves are obtained from those of the symmetric cyclids by circular transformation. It will be convenient to express these circular transformations as linear transformations in the complex plane.

Laplace's Equation will be obtained in curvilinear coördinates corresponding to the families of the system, and it will be shown that in certain special cases an expression for V may be found in the form of Lamé's Products, with an extraneous factor.

The equations of the meridian curves of the symmetric cyclids are

$$\frac{4\rho^2x^2}{\mu - \varepsilon_1} + \frac{4\rho^2r^2}{\mu - \varepsilon_2} + \frac{(x^2 + r^2 - \rho^2)^2}{\mu - \varepsilon_3} - \frac{(x^2 + r^2 + \rho^2)^2}{\mu - \varepsilon_4} = 0, \tag{1}$$

$$\frac{4\rho^2x^2}{\nu - \varepsilon_1} + \frac{4\rho^2r^2}{\nu - \varepsilon_2} + \frac{(x^2 + r^2 - \rho^2)^2}{\nu - \varepsilon_3} - \frac{(x^2 + r^2 + \rho^2)^2}{\nu - \varepsilon_4} = 0. \tag{2}$$

The general linear transformation in the complex plane is

$$x' + r'i = \frac{\alpha + \beta(x + ri)}{\gamma + \delta(x + ri)}, \tag{3}$$

from which follows:

$$x' - r'i = \frac{\alpha_1 + \beta_1(x - ri)}{\gamma_1 + \delta_1(x - ri)}. \tag{4}$$

The constants α, β, γ, δ are not specialized, while α_1, β_1, γ_1, δ_1 are their respective conjugates.

In the previous section, (50) and (51) give the values of x^2 and r^2 in terms of μ and ν.

Let θ be the angle between any meridian plane and the $x'\,y'$ plane, then the relations between y', z', r', θ are

$$y' = r' \cos\theta, \quad z' = r' \sin\theta. \tag{5}$$

Thus the connection has been established between the rectangular coördinates x', y', z' and the curvilinear coördinates μ, ν, θ of the surfaces forming the basis of the present paper.

Laplace's Equation in coördinates μ, ν, θ and a condition for its resolution.

Laplace's Equation in curvilinear coördinates has been given in the previous section, (57).

From (5),

$$\frac{1}{h_1^2} = (D_\mu x')^2 + (D_\mu r')^2, \tag{6}$$

$$\frac{1}{h_2^2} = (D_\nu x')^2 + (D_\nu r')^2, \tag{7}$$

$$\frac{1}{h_3^2} = 1. \tag{8}$$

Assume that V has the form

$$V = \frac{1}{\sqrt{r'}}\, \mathrm{M}\,.\,\mathrm{N}\,.\,\Theta, \tag{9}$$

where M, N, Θ are functions of μ, ν, θ respectively. Applying (8) and (9) to Laplace's Equation in curvilinear coördinates and noticing that h_1 and h_2 are independent of θ, gives

$$h_1 h_2 r'^2 \left[\frac{D_\mu \left(\frac{h_1 r'}{h_2} D_\mu \frac{M}{\sqrt{r'}} \right)}{M} + \frac{D_\nu \left(\frac{h_2 r'}{h_1} D_\nu \frac{N}{\sqrt{r'}} \right)}{N} \right] + \frac{D_\theta^2 \Theta}{\Theta} = 0. \tag{10}$$

Equation (10) may be resolved into two equations, viz.

$$D_\theta^2 \Theta + m^2 \Theta = 0, \tag{11}$$

$$D_\mu \frac{\left(\frac{h_1 r'}{h_2} D_\mu \frac{M}{\sqrt{r'}} \right)}{M} + D_\nu \frac{\left(\frac{h_2 r'}{h_1} D_\nu \frac{N}{\sqrt{r'}} \right)}{N} - \frac{m^2}{h_1 h_2 r'^2} = 0. \tag{12}$$

The general solution of (11) is

$$\Theta = A \cos m\theta + B \sin m\Theta. \tag{13}$$

In order to simplify (12), it is necessary to calculate the value of $\frac{h_1}{h_2}$.

From (3) and (4) come four equations of the form

$$D_\mu x' + iD_\mu r' = \frac{(\beta\gamma - \alpha\delta)(D_\mu x + iD_\mu r)}{(\gamma + \delta(x+ir))^2}. \tag{14}$$

From (6)(7) and (14),

$$\frac{h_2^2}{h_1^2} = \frac{(D_\mu x')^2 + (D_\mu r')^2}{(D_\nu x')^2 + (D_\nu r')^2} = \frac{(D_\mu x)^2 + (D_\mu r)^2}{(D_\nu x)^2 + (D_\nu r)^2}. \tag{15}$$

Equation (15) requires the calculation of four expressions of the type $(D_\mu x)^2$.

From the value of x^2 referred to above

$$\frac{4}{\rho^2} \cdot (D_\mu x)^2 = \frac{(\varepsilon_2 - \varepsilon_3)(\varepsilon_2 - \varepsilon_4)(\varepsilon_3 - \varepsilon_4)}{(\varepsilon_1 - \varepsilon_2)} \cdot \tag{16}$$

$$\frac{\begin{bmatrix}(\varepsilon_1 - \varepsilon_3)\sqrt{(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_4)(\mu - \varepsilon_2)(\mu - \varepsilon_4)(\nu - \varepsilon_1)(\nu - \varepsilon_3)} \\ \pm (\varepsilon_1 - \varepsilon_4)\sqrt{(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)(\mu - \varepsilon_2)(\mu - \varepsilon_3)(\nu - \varepsilon_1)(\nu - \varepsilon_4)}\end{bmatrix}^2}{(\mu - \varepsilon_1)(\mu - \varepsilon_2)(\mu - \varepsilon_3)(\mu - \varepsilon_4)\begin{bmatrix}\sqrt{(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_4)(\mu - \varepsilon_3)(\nu - \varepsilon_3)} \\ \pm \sqrt{(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)(\mu - \varepsilon_4)(\nu - \varepsilon_4)}\end{bmatrix}^4}.$$

Interchanging ε_1 and ε_2 in (16) gives $(D_\mu r)^2$. From these two values an interchange of μ and ν gives $(D_\nu x)^2$ and $(D_\nu r)^2$.

From the four expressions above mentioned,

$$(D_\mu x)^2 + (D_\mu r)^2 =$$

$$\frac{\rho^2}{4} \cdot \frac{\dfrac{(\varepsilon_1 - \varepsilon_3)(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_3)(\varepsilon_2 - \varepsilon_4)(\varepsilon_3 - \varepsilon_4)(\nu - \mu)}{(\mu - \varepsilon_1)(\mu - \varepsilon_2)(\mu - \varepsilon_3)(\mu - \varepsilon_4)}}{\left[\sqrt{(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_4)(\mu - \varepsilon_3)(\nu - \varepsilon_3)} \pm \sqrt{(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)(\mu - \varepsilon_4)(\nu - \varepsilon_4)}\right]^2}, \tag{17}$$

$$(D_\nu x)^2 + (D_\nu r)^2 =$$

$$\frac{\rho^2}{4} \cdot \frac{\dfrac{(\varepsilon_1 - \varepsilon_3)(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_3)(\varepsilon_2 - \varepsilon_4)(\varepsilon_3 - \varepsilon_4)(\mu - \nu)}{(\nu - \varepsilon_1)(\nu - \varepsilon_2)(\nu - \varepsilon_3)(\nu - \varepsilon_4)}}{\left[\sqrt{(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_4)(\mu - \varepsilon_3)(\nu - \varepsilon_3)} \pm \sqrt{(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)(\mu - \varepsilon_4)(\nu - \varepsilon_4)}\right]}. \tag{18}$$

By the aid of (17)(18) equation (15) becomes

$$\frac{h_2^2}{h_1^2} = \frac{-(\nu - \varepsilon_1)(\nu - \varepsilon_2)(\nu - \varepsilon_3)(\nu - \varepsilon_4)}{(\mu - \varepsilon_1)(\mu - \varepsilon_2)(\mu - \varepsilon_3)(\mu - \varepsilon_4)}. \tag{19}$$

Assume two new parameters s and t defined by the equations

$$s=\int\frac{d\mu}{2\sqrt{(\mu-\varepsilon_1)(\mu-\varepsilon_2)(\mu-\varepsilon_3)(\mu-\varepsilon_4)}}, \tag{20}$$

$$t=\int\frac{d\nu}{2\sqrt{(\nu-\varepsilon_1)(\nu-\varepsilon_2)(\nu-\varepsilon_3)(\nu-\varepsilon_4)}}. \tag{21}$$

From (19) . . . (21)

$$\frac{h_2^2}{h_1^2}+\frac{\left(\frac{d\nu}{dt}\right)^2}{\left(\frac{d\mu}{ds}\right)^2}=0. \tag{22}$$

Using (20) . . . (22), it is now possible to write (12) in a new form, viz.

$$\frac{D_s^2M}{M}-\frac{D_t^2N}{N}-\frac{1}{2r'}\left[D_s^2r'-D_t^2r'\right] \\ +\frac{1}{4r'^2}\left[(D_sr')^2-(D_tr')^2\right]\mp\frac{m^2i}{r^2h_1h_2}\cdot\left(\frac{d\mu}{ds}\right)\cdot\left(\frac{d\nu}{dt}\right)=0. \tag{23}$$

From (7),

$$\frac{1}{h_2^2}=(D_\nu x')^2+(D_\nu r')^2=\frac{(D_tx')^2+(D_tr')^2}{\left(\frac{d\nu}{dt}\right)^2}. \tag{24}$$

The last term of (23) is reduced by the use of (22) and (24), thus:

$$\frac{\mp m^2i}{r^2h_1h_1}\left(\frac{d\mu}{ds}\right)\left(\frac{d\nu}{dt}\right)=\frac{m^2}{r'^2}\left[(D_tx')^2+(D_tr')^2\right]. \tag{25}$$

The third and fourth terms of (23) are reduced by the following calculations.

From (16) and three others of the same type, and with the aid of (20) and (21),

$$\frac{(D_\mu x)^2}{(D_\mu r)^2}=\frac{-(\nu-\varepsilon_1)(\nu-\varepsilon_2)(\nu-\varepsilon_3)(\nu-\varepsilon_4)}{(\mu-\varepsilon_1)(\mu-\varepsilon_2)(\mu-\varepsilon_3)(\mu-\varepsilon_4)}=\frac{-\left(\frac{d\nu}{dt}\right)^2}{\left(\frac{d\mu}{ds}\right)^2}, \tag{26}$$

$$\frac{(D_\nu x)^2}{(D_\nu r)^2}=\frac{(\mu-\varepsilon_1)(\mu-\varepsilon_2)(\mu-\varepsilon_3)(\mu-\varepsilon_4)}{(\nu-\varepsilon_1)(\nu-\varepsilon_2)(\nu-\varepsilon_3)(\nu-\varepsilon_4)}=\frac{-\left(\frac{d\mu}{ds}\right)^2}{\left(\frac{d\nu}{dt}\right)^2}. \tag{27}$$

Equations (26) and (27) give

$$(D_s x)^2 + (D_t r)^2 = 0, \tag{28}$$

$$(D_t x)^2 + (D_s r)^2 = 0. \tag{29}$$

The condition for orthogonal curves is

$$D_s x \,.\, D_t x + D_s r \,.\, D_t r = 0. \tag{30}$$

From (3) and (4) come four equations of the following type:

$$2D_s x' = \frac{(\beta\gamma - \alpha\delta)(D_s x + iD_s r)}{(\gamma + \delta(x + ir))^2} + \frac{(\beta_1\gamma_1 - \alpha_1\delta_1)(D_s x - iD_s r)}{(\gamma_1 + \delta_1(x - ir))^2}. \tag{31}$$

Equations (28) . . . (31) give

$$D_s x' = \pm iD_t r', \quad D_t x' = \pm iD_s r', \tag{32}$$

from which follow

$$D_s^2 r' - D_t^2 r' \qquad = 0, \tag{33}$$

$$(D_s r)^2 + (D_t x')^2 = 0. \tag{34}$$

From (25)(33)(34) the final form of (23) is

$$\frac{D_s^2 M}{M} - \frac{D_t^2 N}{N} + (m^2 - \tfrac{1}{4})\frac{[(D_t x')^2 + (D_t r')^2]}{r'^2} = 0. \tag{35}$$

Since the first two terms of (35) are functions of s and t respectively, it follows that the third term must be the sum of two functions, one of s alone and the other of t alone. In that case the equation may be resolved into two ordinary differential equations. Hence the necessary and sufficient condition for the existence of a solution of Laplace's equation in the form

$$V = \frac{1}{\sqrt{r}} . \mathrm{M} . \mathrm{N} . \Theta \tag{36}$$

is the following:

$$\frac{(D_t x')^2 + (D_t r')^2}{r'^2} = H_1(s) + H_2(t). \tag{37}$$

Four cases resulting from the condition.

To determine the constants in the general linear transformation, (3), in such a way that (37) shall be satisfied, is the next important topic in this paper.

By the aid of (3) and (4) the first member of (37) becomes

$$\frac{(D_tx')^2+(D_tr')^2}{r'^2}=\frac{K[(D_tx)^2+(D_tr)^2]}{[A+E(x^2+r^2)+(B+D)x+(B-D)ir]^2}. \tag{38}$$

In (38) the following abbreviations are used:

$$A=\alpha\gamma_1-\alpha_1\gamma,\quad B=\beta\gamma_1-\beta_1\gamma, \tag{39}$$

$$D=\alpha\delta_1-\alpha_1\delta,\quad E=\beta\delta_1-\beta_1\delta, \tag{40}$$

$$K=-4(\alpha\delta-\beta\gamma)(\alpha_1\delta_1-\beta_1\gamma_1). \tag{41}$$

The numerator of (38) may be reduced by noticing that

$$(D_tx)^2+(D_tr)^2=[(D_\nu x)^2+(D_\nu r)^2]\left(\frac{d\nu}{dt}\right)^2. \tag{42}$$

From (18)(21)(42)

$$(D_tx)^2+(D_tr)=$$

$$\frac{\rho^2(\varepsilon_1-\varepsilon_3)(\varepsilon_1-\varepsilon_4)(\varepsilon_2-\varepsilon_3)(\varepsilon_2-\varepsilon_4)(\varepsilon_3-\varepsilon_4)(\mu-\nu)}{[\sqrt{(\varepsilon_1-\varepsilon_4)(\varepsilon_2-\varepsilon_4)(\mu-\varepsilon_3)(\nu-\varepsilon_3)}\pm\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)(\mu-\varepsilon_4)(\nu-\varepsilon_4)}]^2}. \tag{43}$$

From the values of x^2 and r^2, after some reduction,

$$x^2+r^2=-\rho^2\frac{[\sqrt{(\varepsilon_1-\varepsilon_4)(\varepsilon_2-\varepsilon_4)(\mu-\varepsilon_3)(\nu-\varepsilon_3)}\mp\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)(\mu-\varepsilon_4)(\nu-\varepsilon_4)}]}{\sqrt{(\varepsilon_1-\varepsilon_4)(\varepsilon_2-\varepsilon_4)(\mu-\varepsilon_3)(\nu-\varepsilon_3)}\pm\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)(\mu-\varepsilon_4)(\nu-\varepsilon_4)}}. \tag{44}$$

Several new abbreviations are introduced as follows:

$$\overline{A}=\pm\rho(B+D)\sqrt{\frac{(\varepsilon_2-\varepsilon_3)(\varepsilon_2-\varepsilon_4)(\varepsilon_3-\varepsilon_4)}{(\varepsilon_1-\varepsilon_2)}}, \tag{45}$$

$$\overline{B}=\pm\rho(B-D)\sqrt{\frac{(\varepsilon_1-\varepsilon_3)(\varepsilon_1-\varepsilon_4)(\varepsilon_3-\varepsilon_4)}{(\varepsilon_1-\varepsilon_2)}}, \tag{46}$$

$$\overline{C}=(A-E\rho^2)\sqrt{(\varepsilon_1-\varepsilon_4)(\varepsilon_2-\varepsilon_4)}, \tag{47}$$

$$\overline{D}=\pm(A+E\rho^2)\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)}, \tag{48}$$

$$H=\rho^2(\varepsilon_1-\varepsilon_3)(\varepsilon_1-\varepsilon_4)(\varepsilon_2-\varepsilon_3)(\varepsilon_2-\varepsilon_4)(\varepsilon_3-\varepsilon_4). \tag{49}$$

None of the quantities introduced by (39) . . . (41)(45) . . . (49) involve μ or ν. By means of (43) . . . (49) a new form is obtained for (38), viz.

$$\frac{(D_tx')^2+(D_tr')^2}{r'^2}=\frac{H.K.(\mu-\nu)}{\left[\begin{array}{l}\overline{A}\sqrt{(\mu-\varepsilon_1)(\nu-\varepsilon_1)}+\overline{B}\sqrt{(\mu-\varepsilon_2)(\nu-\varepsilon_2)}\\ \quad+\overline{C}\sqrt{(\mu-\varepsilon_3)(\nu-\varepsilon_3)}+\overline{D}\sqrt{(\mu-\varepsilon_4)(\nu-\varepsilon_4)}\end{array}\right]^2}. \tag{50}$$

Let the second member of (50) be denoted by $H.K.\phi(\mu, \nu)$. Then from (37) it follows that values of $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ are to be found such that $\phi(\mu, \nu)$ shall be equal to the sum of two functions, one of μ alone and the other of ν alone. It is evident that the necessary and sufficient condition is

$$D_\nu D_\mu \phi(\mu\nu) = 0. \tag{51}$$

A single term in (51) is $\dfrac{J.\bar{C}\bar{D}}{\Sigma(\mu, \nu)}$, where J and Σ have the following meanings:

$$J = \frac{-3(\mu-\nu)(\varepsilon_3-\varepsilon_4)^2}{2\sqrt{(\mu-\varepsilon_3)(\mu-\varepsilon_4)(\nu-\varepsilon_3)(\nu-\varepsilon_4)}}, \tag{52}$$

$$\begin{aligned}\Sigma(\mu, \nu) = [\bar{A}\sqrt{(\mu-\varepsilon_1)(\nu-\varepsilon_1)} &+ \bar{B}\sqrt{(\mu-\varepsilon_2)(\nu-\varepsilon_2)} \\ &+ \bar{C}\sqrt{(\mu-\varepsilon_3)(\nu-\varepsilon_3)} + \bar{D}\sqrt{(\mu-\varepsilon_4)(\nu-\varepsilon_4)}]^4.\end{aligned} \tag{53}$$

The coefficients of $\bar{A}^2$, $\bar{B}^2$, $\bar{C}^2$, $\bar{D}^2$ in (51) vanish, so that the final form contains only the six terms of the type of $\bar{C}\bar{D}$. These two abbreviations are made:

$$\Gamma(\mu, \nu) = \sqrt{(\mu-\varepsilon_1)(\mu-\varepsilon_2)(\mu-\varepsilon_3)(\mu-\varepsilon_4)(\nu-\varepsilon_1)(\nu-\varepsilon_2)(\nu-\varepsilon_3)(\nu-\varepsilon_2)}, \tag{54}$$

$$\begin{aligned}\Phi(\mu, \nu) = &\bar{A}\bar{B}(\varepsilon_1-\varepsilon_2)^2\sqrt{(\mu-\varepsilon_3)(\mu-\varepsilon_4)(\nu-\varepsilon_3)(\nu-\varepsilon_4)} \\ &+ \bar{A}\bar{C}(\varepsilon_1-\varepsilon_3)^2\sqrt{(\mu-\varepsilon_2)(\mu-\varepsilon_4)(\nu-\varepsilon_2)(\nu-\varepsilon_4)} \\ &+ \bar{A}\bar{D}(\varepsilon_1-\varepsilon_4)^2\sqrt{(\mu-\varepsilon_2)(\mu-\varepsilon_3)(\nu-\varepsilon_2)(\nu-\varepsilon_3)} \\ &\pm \bar{B}\bar{C}(\varepsilon_2-\varepsilon_3)^2\sqrt{(\mu-\varepsilon_1)(\mu-\varepsilon_4)(\nu-\varepsilon_1)(\nu-\varepsilon_4)} \\ &+ \bar{B}\bar{D}(\varepsilon_2-\varepsilon_4)^2\sqrt{(\mu-\varepsilon_1)(\mu-\varepsilon_3)(\nu-\varepsilon_1)(\nu-\varepsilon_3)} \\ &+ \bar{C}\bar{D}(\varepsilon_3-\varepsilon_4)^2\sqrt{(\mu-\varepsilon_1)(\mu-\varepsilon_2)(\nu-\varepsilon_1)(\nu-\varepsilon_2)}.\end{aligned} \tag{55}$$

Equation (51) becomes

$$\frac{(\mu-\nu)\Phi(\mu, \nu)}{\Gamma(\mu, \nu)\Sigma(\mu, \nu)} = 0. \tag{56}$$

The first member of (56) is to vanish for all values of μ and ν, but, from (48), will become infinite for μ or ν equal to ε_1, ε_2, ε_3, ε_4 unless $\Phi(\mu, \nu)$ shall vanish for these values of μ and ν.

There follow then twelve equations of the type

$$\Phi(\varepsilon_1\varepsilon_2) = 0, \tag{57}$$

from which comes, by reference to (55),

$$\bar{A}\bar{B} = \bar{A}\bar{C} = \bar{A}\bar{D} = \bar{B}\bar{C} = \bar{B}\bar{D} = \bar{C}\bar{D} = 0. \tag{58}$$

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Since $\Sigma(\mu, \nu)$ cannot vanish, else ϕ would have no meaning, the following case is excluded,

$$\overline{A} = \overline{B} = \overline{C} = \overline{D} = 0. \tag{59}$$

Equations (58) and (59) give four cases

$$\text{(a)} \quad \overline{A} \neq 0, \quad \overline{B} = \overline{C} = \overline{D} = 0, \tag{60}$$

$$\text{(b)} \quad \overline{B} \neq 0, \quad \overline{C} = \overline{D} = \overline{A} = 0, \tag{61}$$

$$\text{(c)} \quad \overline{C} \neq 0, \quad \overline{D} = \overline{A} = \overline{B} = 0, \tag{62}$$

$$\text{(d)} \quad \overline{D} \neq 0, \quad \overline{A} = \overline{B} = \overline{C} = 0. \tag{63}$$

The conditions given in (60) . . . (63) are not only necessary in order that $D_\nu D_\phi(\mu, \nu)$ may be finite for certain values of μ and ν, but they are also sufficient to make $D_\nu D_\phi(\mu, \nu)$ vanish for all values of μ and ν, as is evident from (55). From (45) . . . (48) the four cases above become

$$\text{(a)} \quad B \neq 0, \quad A = E = B - D = 0, \tag{64}$$

$$\text{(b)} \quad B \neq 0, \quad A = E = B + D = 0, \tag{65}$$

$$\text{(c)} \quad A \neq 0, \quad B = D = A + \rho^2 E = 0, \tag{66}$$

$$\text{(d)} \quad A \neq 0, \quad B = D = A - \rho^2 E = 0. \tag{67}$$

Since A, B, C, D are functions of α, β, γ, δ, it follows that (64) . . . (67) are the conditions under which (37) may exist, and thus make it possible to obtain V in the form desired. In case (a), equation (38), is resolved into the two ordinary differential equations

$$\frac{d^2M}{ds^2} - \left[\frac{(m^2 - \frac{1}{4})\, H.K}{\mu - \varepsilon_1} + n\right] M = 0, \tag{68}$$

$$\frac{d^2N}{dt^2} - \left[\frac{(m^2 - \frac{1}{4})\, H.K}{\nu - \varepsilon_1} + n\right] N = 0. \tag{69}$$

Cases (b)(c)(d) give similar results.

If $E_1(\mu)$, $E_2(\nu)$ be the solutions of any pair of equations of the form of (68)(69), then the value of V is

$$V = \frac{1}{\sqrt{r'}} \left[A \cos m\theta + B \sin m\theta\right] E_1(\mu)\, E_2(\nu). \tag{70}$$

Equation (70) completes the solution of the problem of expressing V in the form of Lamé's Products and an extraneous factor.

Meridian curves corresponding to the four cases.

Having obtained the conditions satisfied by $\alpha, \beta, \gamma, \delta$, the nature of the meridian curves corresponding to the four cases may now be determined. Equations (39)(40) express A, B, D, E in terms of $\alpha, \beta, \gamma, \delta$ and their conjugates.

Separating $\alpha, \beta, \gamma, \delta$ into real and imaginary parts gives

$$\alpha = a + a_0 i, \quad \alpha_1 = a - a_0 i, \quad \gamma = c + c_0 i, \quad \gamma_1 = c - c_0 i, \tag{71}$$

$$\beta = b + b_0 i, \quad \beta_1 = b - b_0 i, \quad \delta = d + d_0 i, \quad \delta_1 = d - d_0 i. \tag{72}$$

It is assumed that the determinant of the linear transformation does not vanish, i. e.

$$\alpha\delta - \beta\gamma \neq 0. \tag{73}$$

It follows from (71)(72) that the two following expressions cannot vanish simultaneously:

$$ad - bc + b_0 c_0 - a_0 d_0, \tag{74}$$

$$a_0 d - b_0 c - b c_0 + a d_0. \tag{75}$$

From (64)(71) . . . (75) the following results for case (a) are obtained:

$$\frac{a}{a_0} = \frac{-b_0}{b} = \frac{c}{c_0} = \frac{-d_0}{d} = m \; (m \text{ real}, \neq 0), \tag{76}$$

$$x' + r'i = \frac{a + b_0 i (x + ri)}{c + d_0 i (x + ri)}. \tag{77}$$

Similarly for case (b),

$$\frac{a}{a_0} = \frac{b}{b_0} = \frac{c}{c_0} = \frac{d}{d_0} = n \; (n \text{ real}, \neq 0), \tag{78}$$

$$x' + r'i = \frac{a + b (x + ri)}{c + d (x + ri)}. \tag{79}$$

An analysis of (77) will show that the surfaces of the system may be obtained from the curves of (1) and (2) by subjecting them to an inversion with regard to a point on the r axis, followed by a translation along that axis and then using it as the axis of rotation. Similarly from (79) it may be shown that the centre of inversion is on the x axis, and the same axis is the axis of revolution. For cases (c) and (d) the equations found are

$$ad + a_0 d_0 - bc - b_0 c_0 = 0, \tag{80}$$

$$a_0 d - a d_0 + b_0 c - b c_0 = 0, \tag{81}$$

$$a_0 c - a c_0 \pm \rho^2 (b_0 d - b d_0) = 0, \tag{82}$$

$$\pm \rho^2 = \frac{a^2 + a_0^2}{b^2 + b_0^2} = \frac{c^2 + c_0^2}{d^2 + d_0^2}. \tag{83}$$

In (82)(83) the upper signs belong to case (a), the lower signs to case (b).

From (83), case (c) exists only when ρ^2 is a positive real quantity. The transformation may be followed by considering its effect on the circle whose radius is ρ and centre is at the origin. The equation of this circle may be written, with ϕ as a variable angle,

$$z = \rho (\cos \phi + i \sin \phi). \tag{84}$$

The transformation is

$$x' + r'i = \frac{\alpha + \beta z}{\gamma + \delta z}, \tag{85}$$

where α, β, γ, δ are subject to the conditions of (80) . . . (82). From (84)(85),

$$x' + ir' = \frac{P + Qi}{S}, \tag{86}$$

where P, Q, S are real and Q has the value

$$\begin{aligned} Q = {} & ca_0 - ac_0 + \rho^2 (b_0 d - b d_0) \\ & + \rho \sin \phi (bc - ad + b_0 c_0 - a_0 d_0) + \rho \cos \phi (b_0 c - b c_0 + a_0 d - a d_0). \end{aligned} \tag{87}$$

But (80) . . . (82) make Q become zero, and (86) gives

$$r' = 0. \tag{88}$$

It follows that in case (c), provided ρ^2 is real and positive, the circle whose radius is ρ is transformed into the axis of revolution. This circle is a circle of symmetry, as may be shown from (1) and (2).

A similar treatment of case (d) shows that it exists only when ρ^2 is a negative real quantity, and that the circle of symmetry whose radius is $\pm \rho i$ is transformed into the axis of revolution.

The four cases may be stated briefly as follows: In case (a) the r axis becomes the axis of revolution, while in case (b) the axis of revolution is unchanged.

In cases (c) and (d) the real circle of symmetry, if such exists, becomes the axis of revolution.

In all cases an inversion may take place with regard to a point on the axis of revolution, followed by a translation along that axis, this inversion of the curves resulting in a space inversion of the surfaces.

The real surfaces obtained.

The meridian curves of (1) and (2) are real if ρ^2, ε_1, ε_2, ε_3, ε_4 are real, also if ρ^2 is pure imaginary, while ε_1, ε_2 are real and ε_3, ε_4 conjugate imaginaries. In the first instance the curves are those which Holzmüller* calls sn. curves, and in the second instance cn. curves, convenient designations which will be retained in the following pages. The transformations in cases (a)(b)(c)(d) are applicable to the sn. curves, while cases (a) and (b) only are applicable to the cn. curves.

The four cases give no new surfaces, for all the surfaces obtained by them may be obtained by the following method: If (1) and (2) represent the sn. curves, then an interchange of ε_1, ε_2, ε_3, ε_4, followed by inversion with regard to a point on the axis of revolution, and a translation along that axis, gives all the surfaces of cases (a)(b)(c)(d).

If the equations represent the cn. curves, then ε_1 and ε_2 may be interchanged, followed by inversion and translation as before, giving the surfaces of cases (a) and (b).

If the surfaces are not real, then cases (a) and (b) only exist, and one or the other of the coördinate axes is the axis of revolution, admitting inversion and translation as before.

Confocal spherical cyclic curves.

Confocal spherical cyclic curves may be defined as the curves on the surface of a sphere obtained by inversion from the general confocal plane cyclic curves.

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*Isogonale Verwandschaften, Holzmüller, S. 256.

It appears from the work of the previous section that if the most general family for which it has now been found that Lamé's Products, with the extraneous factor $\frac{1}{\sqrt{r}}$ can be obtained, be subjected to the inverse transformation, changing meridian planes into concentric spheres, the result is merely confocal cones of the second degree. It will now be shown that the general family of spherical cyclic curves goes over, by the direct transformation—equations (32) . . . (36), Section I—into a family of confocal plane cyclic curves.

The detailed transformations of the previous paper show that one sphere of the family of concentric spheres becomes a plane after the first inversion.

In order to prove the proposition above, consider the general unsymmetrical cyclic curves and let the plane of these curves be the plane derived from a certain sphere of a family of concentric spheres by a portion of what has been called the direct transformation. If the inverse of this transformation be now applied to the plane of the curves, the result is the original sphere, and the general family of confocal spherical cyclic curves.

In applying the transformation which will carry concentric spheres into meridian planes, the first portion will merely reproduce the original curves, while the remainder will be essentially an inversion of the plane of the curves. Consequently the curves finally obtained are plane confocal cyclic curves, for they are the result of a linear transformation of the original curves.

Since it has been shown in this paper that Lamé's Products, with the extraneous factor $\frac{1}{\sqrt{r}}$, can be found only when the cyclic curves in the meridian planes have the axis of revolution as an axis of symmetry, it follows that for a system of conical coördinates in which confocal cyclic curves are cut out on concentric spheres, Lamé's Products can be found only when the cones are of the second order.

Articles by Wangerin and Haentzschel.

In addition to the article previously mentioned, Wangerin has written another article* (treating of surfaces of revolution) in which the extraneous factor is at first undetermined. He obtains the sufficient conditions under which (37) may exist, and also obtains $\frac{1}{\sqrt{r}}$ as the form of the extraneous factor. That his conditions are also necessary follows from a discussion by Boehm.*

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*Reduction der Potentialgleichung, Wangerin. Monatsberichte der Kgl. Akademie der Wissenschaften zu Berlin, 1878.

Corresponding to these conditions, Wangerin obtains surfaces of the fourth degree whose meridian curves are the sn. and cn. curves. These curves may be obtained by equating the real and imaginary parts of the following equation:

$$x + ri = f(t + m), \tag{89}$$

where f is either sn. or cn.

The curves obtained by using tn. or dn. are of the same form as the sn. curves.

Wangerin states that the most general surfaces of revolution for which Lamé's Products, with an extraneous factor, exist, are those whose meridian curves are obtained from the curves above by an inversion with respect to a point on the axis of revolution. These surfaces may be shown to be identical with those obtained in the preceding pages.

Haentzschel† treats some of the same topics as Wangerin but obtains surfaces of revolution of the thirty-second degree, which will not be considered here. He also treats at some length a special form of his most general solution, viz. a linear transformation of the Weierstrass $\wp$ function, where the coefficients are subject to restrictions. (Equation (89) with f written as $\wp$ gives the fundamental curves.)

The resulting surfaces of the fourth degree are identical with those of the present paper. He also obtains surfaces which he states to be of the eighth degree, but these are in fact of only the fourth degree, and of the same type as above.

He has overlooked the fact that the restrictions which he mentions will give a zero value to several coefficients in equations (31)(32)(36) on page 23 of his book.

HARVARD UNIVERSITY, *Dec.*, 1897.

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*Boehm, Reduction Partieller Differentialgleichungen, Leipzig, 1896.

†Reduction der Potentialgleichung, E. Haentzschel, Berlin, 1893.

Elementary Proof of Cauchy's Theorem.

BY ARTHUR LATHAM BAKER.

Denote the sides of a right triangle (Argand Diagram) by x, iy, z. Multiply each of these by the complex number w (plane vector) giving $wz = wx + wiy$. Hence the proposition: *If on the three sides of a right triangle, similar and similarly placed triangles be constructed, then the sum of the corresponding sides (considered as vectors) of the leg triangles is equal to the corresponding side of the hypothenuse triangle.*

At the limit this becomes

$$dW = wdz = wdx + widy = w(dx + idy).$$

Hence *the change in the function W is the same whether z follows the elementary paths dx, idy, or the resultant of these, dz.*

Hence $W = \int dW$, the sum of all these changes in W due to changes in z, will be the same whether z follows the elemental paths Σdx, Σidy or resultants of these, so long as in the deformation of the path of z into Σdx and Σidy we do not pass over any point in which $w = \infty$.

Hence W *always attains the same value for the same value of z, whatever the path of z, provided no point where $w = \infty$ is enclosed between the paths.*

Since we can take the end of the path of z as near the beginning as we choose, or coincident with it, we can say:

(Cauchy's Theorem) $W = \int dW = \int wdz$ *taken around a closed curve enclosing no point where $w = \infty$ is zero.*

UNIVERSITY OF ROCHESTER.

Invariants of the General Linear Differential Equation and their Relation to the Theory of Continuous Groups.

BY CHARLES L. BOUTON.

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In the course of the last three decades Lie has created a general invariant theory for all continuous transformation groups which can be defined by differential equations. This far-reaching theory had many precursors, and among these Cayley's invariant theory has played a particularly important role in the mathematical development of our century. The deformation theory of surfaces, founded by Gauss and Minding, may also, as Lie has shown, be regarded as the invariant theory of a group, of an *infinite* group. On the other hand the theory of curvature of Euler and Monge is the invariant theory of the *finite* group of motions.

In the years 1870–72 Lie founded an invariant theory of the general group of all Contact Transformations, and at the same time, of certain infinite groups which he termed "Funktionengruppen." A development of these general investigations was his Theory of Transformation Groups (1874), and his General Invariant Theory of all Transformation Groups, the latter published in the years 1882–1884.

Laguerre and Brioschi in 1879 gave certain methods of finding invariants of linear differential equations; and Halphen, in his prize essay of 1882, gave a general method for finding such invariants, and applied it in the simpler cases. None of these mathematicians, however, refer to the work of an earlier writer on this subject, Cockle, who, as early as 1862, had found invariants of the general equation. The idea that all of these researches may be treated by the invariant theory of infinite groups belongs to Lie.

It is proposed to give in the following paper the derivation, according to Lie's method of continuous groups, of the invariants of the general linear differential equation of the n^{th} order.

The first mention of invariants of the linear differential equation seems to have been by Cockle in 1862. For the next fifteen or twenty years Cockle wrote at intervals on this subject. An account of his work will be found in Chapter 1. It may be thought that this review is rather more detailed than the importance of his results warrants, especially when compared with the amount of space given to the work of other writers. This has been done for two reasons. Cockle's investigations were given in a considerable number of comparatively short papers, and are therefore much more inaccessible than those of other writers on this subject, who gave their results in one or two memoirs. It therefore seemed desirable to bring the methods and results of this *first* writer together in one place. A second reason was that Cockle's examples, involving as they do both invariants and covariants for transformation of one variable at a time, seemed to be particularly well suited for illustrating the application of Lie's methods in the simpler cases, before taking up the general case. Cockle seems to have used only finite transformations, and to have actually found invariants for transformation of only *one* of the variables at a time. Then Laguerre* and Brioschi* in 1879 found certain invariants for transformation of both variables. Laguerre's method of finding "semi-invariants" for transformation of the dependent variable alone, is identical with the method given by Cockle nine years earlier. Laguerre made use of his invariants to remove the two terms of order next to the highest from the equation. Halphen in 1882† and 1883‡ carried the theory still further, and made most important applications of it. In 1884 Lie‖ called attention to the fact that these invariants may all be found by his methods as the invariants of a certain infinite group. Forsyth§ in 1888 gave *all* the invariants and covariants of the equation in a *normal form*, together with a method of finding them for a form of the equation in which the term of order next to the highest is removed. For this latter form of the equation he gives the first five relative invariants. Forsyth's method is one which makes use of *infinitesimal transformations*, and the formulæ he deduces for the increments of the quantities involved are exactly the same as those which must be used in the method of continuous groups. Compare, for example, Forsyth's formulæ (12) and (13) (l. c., pp. 395, 397), with the formula (36) of this paper. In the latter part of his memoir Forsyth deduces partial differential equations of which his invariants are solutions. All of these equations (excepting those which have to do with the variables of the adjoined equation, which is not here considered) may be derived from the equations to be given later, but it has not been thought worth while to point out the identity in each case.

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* Comptes Rendus, t. 88 (1879), pp. 116–119; pp. 224–227.

In the following article, after reviewing Cockle's methods and results in Chapter 1, Lie's methods are used in Chapter 2 to find all of Cockle's results, and to verify some of the statements which he made without proof. Then in Chapters 3, 4, 5 the linear partial differential equations, of which the invariants are solutions, are deduced by Lie's methods. This is done for three cases, viz. for

1. The non-specialized equation.
2. The case in which the second term of the equation is removed.
3. Forsyth's normal form.

The transformations used in each case are the most general which leave the chosen form of equation unchanged. For the first two cases the general equations have been solved for a few special cases of the constants involved. For case 3 the complete solution in explicit form is given, it being then comparatively easy to integrate the equations.

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* Bull. Soc. Math. de France, t. 7 (1879), pp. 105–108.

† Mémoires des Savants Etrangeres, t. 28, 2. series, 300 pp.

‡ Acta Math., t. 3 (1883), pp. 325–380.

‖ Math. Anal., Bd. 24 (1884), p. 573.

§ Phil. Trans., 1888, I, pp. 377–489.

It is believed that the linear differential equations of cases 1 and 2, namely, the equations numbered (30) and (40), are here given for the first time.

CHAPTER 1.

Cockle's Work.

§1. The earliest notice which I have been able to find of the fact that there are functions of the coefficients $a_1, \ldots, a_n$ of the homogeneous linear differential equation

$$\frac{d^n y}{dx^n} + na_1 \frac{d^{n-1}y}{dx^{n-1}} + \frac{n(n-1)}{2} a_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_n y = 0, \tag{1}$$

(where $a_1, \ldots, a_n$ are functions of x alone), which remain invariant for changes of the variables which do not change the form or order of the equation, is in an article by Sir James Cockle on "The Correlations of Analysis," dated November, 1862.* Cockle there shows that when, in the general equation (1), y is subjected to the transformation, $y = v \,.\, u(x)$, where v is the new dependent variable, that

$$a_1^2 - a_2 + a_1' \quad \text{and} \quad 2a_1^3 - 3a_1 a_2 + a_3 - a_1''$$

are absolute invariants. The method of proof was to express the new coefficients $A_1, \ldots, A_n$ in terms of $a_1, \ldots, a_n$, u, and the derivatives of u, and then to eliminate u and its derivatives between the resulting equations. In 1870† Cockle calls such invariant functions as the above *criticoids*, these two being respectively the *quadricriticoid* and *cubicriticoid*. The method of elimination may be used in obtaining criticoids of higher degree, but this method has its disadvantages, namely, "want of directness and generality, and a failure of proof of the existence of criticoids other than those actually obtained or deducible from them by combination" (l. c., p. 202). Then, on pp. 209–210 of this same memoir, Cockle gives the method which he regards as the most satisfactory and final, for he seems to have given no other methods. This method "gives by a direct process results perfectly general, and makes manifest the existence of criticoids of all degrees." The process consists in eliminating the *second term* of the differential equation (1) by the substitution $y = ve^{-\int a_1 dx}$. The coefficients of the transformed equation divided by their respective binomial coefficients, viz. $B_2, B_3, \ldots, B_n$, are then $n-1$ primary criticoids of degree 2 to n respectively. The *quarticriticoid* is

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* Phil. Mag., IV Series, vol. 24, p. 532, §2.

† Phil. Mag., vol. 39, p. 202.

$$a_4 - 4a_1 a_3 - 3a_2^2 + 12a_1^2 a_2 - 6a_1^4 - a_1'''.$$

The following is then Cockle's final method for finding criticoids, carried as far as finding the first three: In (1) substitute $y = ve^{-\int a_1 dx}$. Then we have

$$\frac{d^m y}{dx^m} = \sum_0^m{}_s \frac{m!}{(m-s)!\, s!} \frac{d^s\left(e^{-\int a_1 dx}\right)}{dx^s} \frac{d^{m-s} v}{dx^{m-s}},$$

and we find that the transformed equation becomes

$$\frac{d^n v}{dx^n} + \frac{n(n-1)}{2!} B_2 \frac{d^{n-2} v}{dx^{n-2}} + \frac{n(n-1)(n-2)}{3!} B_3 \frac{d^{n-3} v}{dx^{n-3}} + \ldots + B_n v = 0, \qquad (1')$$

where

$$B_2 = a_2 - a_1^2 - a_1', \quad B_3 = a_3 - 3a_2 a_1 + 2a_1^3 - a_1'',$$
$$B_4 = a_4 - 4a_3 a_1 + 6a_2 a_1^2 - 6a_2 a_1' - 3a_1^4 + 6a_1^2 a_1' + 3a_1'^2 - a_1''',$$

or

$$B_4 = a_4 - 4a_3 a_1 + 12a_2 a_1^2 - 3a_2^2 - 6a_1^4 - a_1''' + 3B_2^2.$$

These functions B_k of the coefficients of the equation (1) are then invariant for any transformation of the dependent variable y, of the form $y = v \cdot u(x)$, where u is an arbitrary function of x. Cockle calls these functions B_k the *primary criticoids.*[*]

It is easy to show that we have in general (although Cockle nowhere explicitly mentions this),

$$B_k = e^{\int a_1 dx} \sum_0^k{}_s \frac{k!}{(k-s)!\, s!} a_{k-s} \frac{d^s\left(e^{-\int a_1 dx}\right)}{dx^s}.$$

In a paper on "Hyperdistributives"[†] Cockle proves that, if we write

$$\theta_m(a) = \frac{d^m \log a}{dx^m},$$

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[*] Phil. Mag., vol. 39 (1870), p. 210, "On Criticoids."

[†] Phil. Mag., vol. 43 (1872), p. 300.

and if after the differentiation we write

$$\frac{1}{a}\,\frac{d^r a}{dx^r} = a_r,$$

and call the resulting function $\theta_m(a_1, a_2, \ldots\ldots, a_m)$, then

$$\theta_m(a_1, \ldots\ldots, a_m) - \frac{d^{m-1}a_1}{dx^{m-1}}$$

is the entire criticoidal function of the m^{th} order.

In another paper* it is stated that the derivative of a criticoid with respect to the independent variable x is a criticoid.

On page 203 of the article in volume 39, Cockle states that he did not attempt to employ an *operative symbol.* By "operative symbol" he means a differential operator which, when applied to the criticoid, reduces it to zero. The operative symbol is of the nature of a partial differential equation, which the function it reduces to zero satisfies. An example of such a symbol, taken from one of Cockle's earlier papers, will be given later.

§2. After having found in 1862 the two differential invariants mentioned above, Cockle, in 1864,† gave a *differential covariant* for the equation of the third order,

$$\frac{d^3y}{dx^3} + 3a_1\frac{d^2y}{dx^2} + 3a_2\frac{dy}{dx} + a_3 y = 0.$$

It is shown by direct computation that

$$B_2\frac{d^2y}{dx^2} + [2B_2a_1 - B_3 - B_2']\frac{dy}{dx} + [B_2a_2 - (B_3 + B_2')\,a_1]\,y,$$

where B_2 and B_3 have the values already given, is a *relative differential covariant* for the substitution $y = v\,.\,u(x)$. The new function is equal to the old multiplied by u. Cockle calls this covariant the "differential Hessian," from its resemblance to a Hessian of an algebraic form. He seems to have found it by trial, for he says (p. 228): "I first deduced the differential Hessian by seeking an expression for which the coefficients of transformation and of substitution should be the same, and I afterward found that an operator, to wit Δ, reduced it to zero." This operative symbol Δ is

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* Phil. Mag., vol. 27 (1864), p. 227.

† Phil. Mag., vol. 27, pp. 225–228.

$$\Delta = \delta - \frac{d}{da_1} - 2a_1\frac{d}{da_2} - 3a_2\frac{d}{da_3},$$

or as it is given in a later paper,*

$$\Delta = \delta - \left[a_0\frac{d}{da_1}\right] - 2\left[a_1\frac{d}{da_2}\right] - 3\left[a_2\frac{d}{da_3}\right],$$

where a_0 is a coefficient multiplied into $\frac{d^3y}{dx^3}$ to make the given differential equation homogeneous in a. Here δ is defined by means of the equation

$$\delta\frac{d^my}{dx^m} = m\frac{d^{m-1}y}{dx^{m-1}}.$$

The symbols in the brackets are supposed to be commutative with $\frac{d}{dx}$, so that we have, for example,

$$\left[a_1\frac{d}{da_2}\right](a_1a_2') = a_1\left[a_1\frac{d}{da_2}\right]a_2' = a_1\left[a_1\frac{d}{da_2}\right]\frac{da_2}{dx} = a_1\frac{d}{dx}\left[a_1\frac{d}{da_2}\right]a_2$$
$$= a_1\frac{da_1}{dx} = a_1a_1'.$$

It will be useful, in view of comparisons to be made later, to express this symbol Δ in terms of partial differential coefficients. Since

$$\left[a_{i-1}\frac{d}{da_i}\right]a_i^{(k)} = \frac{d^k}{dx^k}\left[a_{i-1}\frac{d}{da_i}\right]a_i = a_{i-1}^{(k)},$$

and

$$\left[a_{i-1}\frac{d}{da_i}\right](a_i^{(k)})^m = m\,(a_i^{(k)})^{m-1}\left[a_{i-1}\frac{d}{da_i}\right]a_i^{(k)} = m\,(a_i^{(k)})^{m-1}a_{i-1}^{(k)},$$

we have

$$\left[a_{i-1}\frac{d}{da_i}\right] = \sum_0^k a_{i-1}^{(k)}\frac{\partial}{\partial a_i^{(k)}}, \qquad (i = 1,\ 2,\ 3)$$

and

$$\delta = \sum_1^m my^{(m-1)}\frac{\partial}{\partial y^{(m)}}.$$

Hence the symbol Δ is equivalent to

$$\sum_1^m my^{(m-1)}\frac{\partial}{\partial y^{(m)}} - \sum_0^k a_0^{(k)}\frac{\partial}{\partial a_1^{(k)}} - 2\sum_0^k a_1^{(k)}\frac{\partial}{\partial a_2^{(k)}} - 3\sum_0^k a_2^{(k)}\frac{\partial}{\partial a_3^{(k)}}.$$

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* Phil. Mag., vol. 28 (1864), pp. 205–206.

The upper limits in the different summations are to be chosen large enough so that a farther increase would yield only zero when the operator is applied to the function in question. For the covariant under consideration these developments need only be carried as far as follows:

$$y\frac{\partial}{\partial y'} + 2y'\frac{\partial}{\partial y''} - \frac{\partial}{\partial a_1} - 2a_1\frac{\partial}{\partial a_2} - 2a_1'\frac{\partial}{\partial a_2'} - 3a_2\frac{\partial}{\partial a_3}.$$

We readily verify that this operator reduces the covariant to zero.

The foregoing differential covariant is not the simplest, as Cockle showed in a paper of August, 1865.* Using the notation

$$a_0\frac{d^my}{dx^m} + ma_1\frac{d^{m-1}y}{dx^{m-1}} + \frac{m(m-1)}{2}a_2\frac{d^{m-2}y}{dx^{m-2}} + \dots + a_my \equiv y_m,$$

and designating the differential covariants as *covaroids*, he there states, without proof, that "the functions y_n, y_m, and $\frac{d^py_q}{dx^p}$ are covaroids for all values of m, n, p, and q, and consequently any functions of B_2, B_3, and the higher criticoids, and also of any number of values of y_m and of $\frac{d^py_q}{dx^p}$ are covaroids, and *vice versa*. Accordingly the covaroid which I have called the differential Hessian may, after a slight modification, be put under the form $B_2y_2 - (B_3 + B_2')y_1$, and is only one of an infinite number of quadricovaroids." This is a very important statement, especially the *vice versa*, as that indicates that Cockle had found, or thought he had found, the complete solution of the problem of finding *all* the covariants (and hence all the invariants) of a homogeneous linear differential equation of the n^{th} order, for a transformation of the dependent variable of the form $y = v \cdot u(x)$. This was in 1865. As no proof of this statement is given, we must here regard it as an induction, and examine its truth later.

§3. In 1875† Cockle gave a relative invariant of the general equation (1) for change of the independent variable. The first four pages of the article are devoted to finding the formulæ for change of the independent variable, and the method used, although ingenious, appears very artificial and cumbrous. Having this formula, the work of finding the invariant, or *differential criticoid*, is comparatively short. The result is that if in (1) the independent variable be changed from x to t, where $t = \phi(x)$, so that (1) becomes

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* Phil. Mag., vol. 30, pp. 347–348.

† Phil. Mag., vol. 50, p. 440.

$$\frac{d^n y}{dt^n} + nA_1 \frac{d^{n-1}y}{dt^{n-1}} + \frac{n(n-1)}{2!} A_2 \frac{d^{n-2}y}{dt^{n-2}} + \dots + A_n y = 0,$$

where $A_1, \dots, A_n$ are functions of t alone, that then

$$\left[\frac{dA_1}{dt} + \frac{3n-1}{2(n-1)} A_1^2 - \tfrac{3}{2} \cdot \frac{n-1}{n-2} A_2\right] dt^2 = \left[\frac{da_1}{dx} + \frac{3n-1}{2(n-1)} a_1^2 - \tfrac{3}{2} \cdot \frac{n-1}{n-2} a_2\right] dx^2,$$
$$n > 2,$$

"and either side is a differential criticoid which we may term a differential quadricriticoid." Since y remains unchanged, this amounts to saying that

$$\left[a_1' + \frac{3n-1}{2(n-1)} a_1^2 - \frac{3(n-1)}{2(n-2)} a_2\right] \frac{1}{y'^2}, \qquad n > 2$$

is an absolute covariant for the transformation $t = \phi(x)$. "The differential varies from the ordinary criticoid in this, that the corresponding coefficients of the former contain n, the order of the differential equation, while those of the latter are free from n."

I cannot find that Cockle has published any other invariant for transformation of the independent variable, nor any for the simultaneous transformation of both dependent and independent variables. Harley* mentions, however, that Cockle had in a letter suggested the possibility of the existence of such invariants.

§4. *Resumé of Cockle's Results.*

In 1862 Cockle gave in explicit form two invariants of the general equation (1) for transformation of the dependent variable alone. In 1870 he gave a method of finding a system of invariants, for change of the dependent variable, from which all others may be derived by differentiation, although he does not prove this system to be *complete*, merely mentioning that other invariants may be obtained by differentiating those already found. In 1864 he gave a covariant for the equation of the third order, and in 1865 he stated how all the covariants of the general equation may be found for transformation of the dependent variable alone. In 1875 Cockle gave a single invariant of the general equation for transformation of the independent variable alone. In all of his articles he uses only *finite* transformations, and in the main his methods seem to have been elimination methods, although it is not always clear how he first obtained his results.

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* Royal Soc. Proc., vol. 38 (1884), p. 57.

CHAPTER 2.

Cockle's Results obtained by Lie's Methods.

§5. It will now be shown how Cockle's results would be obtained by Lie's method of continuous groups. The outline of the method as applied to this particular problem was given by Lie in 1884.* The following derivation follows that outline, and is made rather more general than is necessary to obtain the few invariants actually found by Cockle, so that the truth of some of his statements may be examined.

Take first the case in which the dependent variable alone is transformed. The variables x and y are to be transformed by means of

$$x_1 = x, \quad y_1 = y\,\psi(x),$$

where x_1, y_1 are the transformed variables, and ψ is an arbitrary analytic function of x alone. All of these transformations evidently form an infinite continuous group defined by differential equations.

We get the infinitesimal transformation by writing $\psi(x) = 1 + \phi(x)\,.\,\delta t$; then $\delta x = 0$, $\delta y = y\,\phi(x)\,.\,\delta t$, and the symbol of the infinitesimal transformation is

$$\phi(x)\,y\,\frac{\partial f}{\partial y}\,.$$

We are to subject the equation

$$y^{(n)} + na_1y^{(n-1)} + \frac{n(n-1)}{2!}\,a_2y^{(n-2)} + \ldots\ldots + a_ny = 0 \qquad (2)$$

to this transformation, and find functions of the coefficients $a_1, \ldots\ldots, a_n$ and their derivatives which are invariant. To this end we must compute the increments of $a_1, \ldots\ldots, a_n$, and of the various derivatives, and form the *extended*

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* Math. Annal., Band 24, p. 578.

transformation. We have

$$\delta y^{(k)} = \delta \frac{dy^{(k-1)}}{dx} = \frac{d}{dx}\delta y^{(k-1)} - y^{(k)}\frac{d\delta x}{dx} = \frac{d}{dx}\delta y^{(k-1)}$$

$$= \frac{d^2}{dx^2}\delta y^{(k-2)} = \ldots\ldots = \frac{d^k}{dx^k}(\delta y),$$

$$\delta y^{(k)} = \frac{d^k}{dx^k}[\phi(x)y]\,\delta t = \sum_{m=0}^{k}\frac{k!}{(k-m)!\,m!}\phi^{(m)}y^{(k-m)}\,\delta t, \tag{3}$$

by Euler's theorem for the differentiation of a product. Let us take the variation of (2):

$$\delta y^{(n)} + na_1\,\delta y^{(n-1)} + \frac{n(n-1)}{2!}a_2\delta y^{(n-2)} + \ldots\ldots + a_n\delta y$$
$$+ ny^{(n-1)}\delta a_1 + \frac{n(n-1)}{2}y^{(n-2)}\delta a_2 + \ldots\ldots + ny'\,\delta a_{n-1} + y\delta a_n = 0.$$

On substituting the values of $\delta y^{(n-k)}$ as given by (3), this becomes

$$\phi\delta t\sum_{k=0}^{n}\frac{n!}{(n-k)!\,k!}a_k y^{(n-k)} + \sum_{k=1}^{n}\frac{n!}{(n-k)!\,k!}y^{(n-k)}\delta a_k$$
$$+\sum_{k=0}^{n-1}\frac{n!}{(n-k)!\,k!}a_k\sum_{m=1}^{n-k}\frac{(n-k)!}{(n-k-m)!\,m!}\phi^{(m)}y^{(n-k-m)}\,\delta t = 0.$$

Here and in what follows we must understand $a_0 = 1$. Making use of (2) the first sum vanishes. Then changing the letters and order of summation in the last double sum, the preceding equation becomes

$$\sum_{k=1}^{n}\frac{n!}{(n-k)!\,k!}y^{(n-k)}\left[\delta a_k + \sum_{m=0}^{k-1}\frac{k!}{(k-m)!\,m!}\phi^{(k-m)}a_m\,\delta t\right] = 0.$$

This equation must subsist identically for all values of $y^{(n-k)}$, and hence we must have

$$\delta a_k = -\sum_{m=0}^{k-1}\frac{k!}{(k-m)!\,m!}\phi^{(k-m)}a_m\,\delta t = -\sum_{i=1}^{k}\frac{k!}{(k-i)!\,i!}a_{k-i}\,\phi^{(i)}\,\delta t. \tag{4}$$

$$(k = 1,\ 2,\ \ldots\ldots,\ n)$$

We require not only the increments of $a_1, \ldots, a_n$, but also the increment of $a_k^{(s)} = \frac{d^s a_k}{dx^s}$, for $k = 1, 2, \ldots, n$, and all values of s. We have, as in deducing (3),

$$\delta a_k^{(s)} = \delta\left(\frac{da_k^{(s-1)}}{dx}\right) = \frac{d}{dx}\left(\delta a_k^{(s-1)}\right) = \ldots = \frac{d^s}{dx^s}(da_k),$$

whence

$$\delta a_k^{(s)} = -\sum\limits_0^{k-1}{}_m \frac{k!}{(k-m)!\ m!} \frac{d^s}{dx^s}\left(\phi^{(k-m)} a_m\right)\delta t. \tag{5}$$

We have further

$$\frac{d^s}{dx^s}\left(\phi^{(k-m)} a_m\right) = \sum\limits_0^{s}{}_j \frac{s!}{(s-j)!\,j!} \phi^{(k-m+j)} a_m^{(s-j)} = \sum\limits_{k-m}^{s+k-m}{}_i \frac{s!\ a^{(s+k-m-i}\phi^{(i)}}{(s+k-m-i)!\ (i-k+m)!},$$

and consequently,

$$\left.\begin{aligned} \delta a_k^{(s)} &= -\sum\limits_0^{k-1}{}_m \sum\limits_{k-m}^{s+k-m}{}_i \frac{k!\ s!\ a_m^{(s+k-m-i)}\phi^{(i)}\,\delta t}{(k-m)!\ m!\ (s+k-m-i)!\ (i-k+m)!}, \\ \delta a_k^{(s)} &= -\sum\limits_1^{k}{}_m \sum\limits_{m}^{s+m}{}_i \frac{k!\ s!\ a_{k-m}^{(s+m-i)}\phi^{(i)}\,\delta t}{(k-m)!\ m!\ (s+m-i)!\ (i-m)!}. \end{aligned}\right\} \tag{6}$$

$$(k = 1, 2, \ldots; n;\ s = 0, 1, 2, \ldots)$$

These formulæ include (4) as a special case.

§6. Before considering the general case, let us investigate whether there are any invariants involving only a_1 and a_2 and their derivatives, and if so, how many. (6) gives:

$$\delta a_1 = -\phi'\,\delta t, \quad \delta a_1^{(s)} = -\phi^{(s+1)}\,\delta t, \quad \delta a_2 = -(2a_1\phi' + \phi'')\,\delta t,$$

$$\delta a_2^{(s-1)} = -\left[2\sum\limits_1^{s}{}_i \frac{(s-1)!\ a_1^{(s-i)}}{(s-i)!\ (i-1)!}\phi^{(i)} + \phi^{(s+1)}\right]\delta t.$$

Hence the extended transformation is

$$\phi y \frac{\partial f}{\partial y} - \sum\limits_0^{\sigma}{}_s \phi^{(s+1)} \frac{\partial f}{\partial a_1^{(s)}} - \sum\limits_1^{\sigma}{}_s \left\{2\sum\limits_1^{s}{}_i \frac{(s-1)!}{(s-i)!\ (i-1)!} a_1^{(s-i)}\phi^{(i)} + \phi^{(s+1)}\right\} \frac{\partial f}{\partial a_2^{(s-1)}}.$$

Any invariant must be a solution of the partial differential equation obtained by equating this transformation to zero, and that whatever the function ϕ may be. This condition is therefore equivalent to the following set of equations obtained by equating the coefficients of the separate derivatives of ϕ to zero:

$$
\begin{aligned}
0 &= \frac{\partial f}{\partial y}\,, \\
0 &= \frac{\partial f}{\partial a_1} + 2a_1\frac{\partial f}{\partial a_2} + 2\sum_{2}^{\sigma}{}_{s}\, a_1^{(s-1)}\frac{\partial f}{\partial a_2^{(s-1)}}\,, \\
0 &= \frac{\partial f}{\partial a_1'} + \frac{\partial f}{\partial a_2} + 2\sum_{2}^{\sigma}{}_{s}\,(s-1)\,a_1^{(s-2)}\frac{\partial f}{\partial a_2^{(s-1)}}\,, \\
0 &= \frac{\partial f}{\partial a_1''} + \frac{\partial f}{\partial a_2'} + 2\sum_{3}^{\sigma}{}_{s}\,\frac{(s-1)!}{(s-3)!\;2!}\,a_1^{(s-3)}\frac{\partial f}{\partial a_2^{(s-1)}}\,, \\
0 &= \frac{\partial f}{\partial a_1'''} + \frac{\partial f}{\partial a_2''} + 2\sum_{4}^{\sigma}{}_{s}\,\frac{(s-1)!}{(s-4)!\;3!}\,a_1^{(s-4)}\frac{\partial f}{\partial a_2^{(s-1)}}\,, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
0 &= \frac{\partial f}{\partial a_1^{(\sigma-1)}} + \frac{\partial f}{\partial a_2^{(\sigma-2)}} + 2a_1\frac{\partial f}{\partial a_2^{(\sigma-1)}}\,, \\
0 &= \frac{\partial f}{\partial a_1^{(\sigma)}} + \frac{\partial f}{\partial a_2^{(\sigma-1)}}\,.
\end{aligned}
$$

Now, according to Lie's theory,* this system of equations must be a *complete* system, of which the solutions are the invariants sought. It is also easy to verify directly by combining these equations that they form a complete system. Neglecting the first equation, which shows only that the invariant cannot involve y, the number of the variables is $\sigma + 1 + \sigma = 2\sigma + 1$. The number of equations is $\sigma + 1$, and hence there are exactly $(2\sigma + 1) - (\sigma + 1) = \sigma$ independent solutions, since the equations are obviously linearly independent. One solution is found to be

$$I_2 = a_2 - a_1^2 - a_1'.$$

This is Cockle's quadricriticoid. Since x is not transformed, $\dfrac{dI_2}{dx}$ must be invariant, for if u and v are any two invariants of the linear differential equation, then $\dfrac{dv}{du}$ is also invariant.† Hence the σ solutions of the foregoing system may be chosen as

$$I_2,\ \frac{dI_2}{dx},\ \frac{d^2I_2}{dx^2},\ \ldots\ldots,\ \frac{d^{\sigma-1}I_2}{dx^{\sigma-1}};$$

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* Math. Annal., Band 24.

† As a proof of this, for the general case is given later, a special proof is not given here. See §16.

for these invariants are linearly independent of one another, and involve only the variables of the system. We may say then that there is only one *essential* invariant involving only a_1 and a_2 and their derivatives to any order, as all others may be derived from this one by successive differentiation with regard to x. (Compare Cockle's statement that "the differentials of critical functions with regard to the independent variable are critical.")

§7. Next consider the invariants involving only a_1, a_2, a_3 and their derivatives. We have, in addition to the values already given,

$$\delta a_3 = -(3a_2\phi' + 3a_1\phi'' + \phi''')\,\delta t,$$

$$\delta a_3^{(s-2)} = -\Big[3\sum_0^{s-2}{}^{i}\frac{(s-2)!}{(s-2-i)!\;i!}a_2^{(s-2-i)}\phi^{(i+1)}$$
$$+\,3\sum_0^{s-3}{}^{i}\frac{(s-2)!}{(s-2-i)!\;i!}a_1^{(s-3-i)}\phi^{(i+2)}+\phi^{(s+1)}\Big]\,\delta t.$$

Forming now the extended transformation as before, and writing the first three of the resulting equations, we have

$$0=\frac{\partial f}{\partial a_1}+2a_1\frac{\partial f}{\partial a_2}+3a_2\frac{\partial f}{\partial a_3}+2\sum_2^{\sigma}{}^{s}a_1^{(s-1)}\frac{\partial f}{\partial a_2^{(s-1)}}+3\sum_3^{\sigma}{}^{s}a_2^{(s-2)}\frac{\partial f}{\partial a_3^{(s-2)}},$$

$$0=\frac{\partial f}{\partial a_1'}+\quad\frac{\partial f}{\partial a_2}+3a_1\frac{\partial f}{\partial a_3}+2\sum_2^{\sigma}{}^{s}(s-1)\,a_1^{(s-2)}\frac{\partial f}{\partial a_2^{(s-1)}}$$
$$+\,3\sum_3^{\sigma}{}^{s}\big[(s-2)\,a_2^{(s-3)}+a_1^{(s-2)}\big]\frac{\partial f}{\partial a_3^{(s-2)}},$$

$$0=\frac{\partial f}{\partial a_1''}+\quad\frac{\partial f}{\partial a_2'}+\quad\frac{\partial f}{\partial a_3}+2\sum_3^{\sigma}{}^{s}\frac{(s-1)(s-2)}{2}a_1^{(s-3)}\frac{\partial f}{\partial a_2^{(s-1)}}$$
$$+\,3\sum_3^{\sigma}{}^{s}\Big[\frac{(s-2)(s-3)}{2}a_2^{(s-4)}+(s-2)\,a_1^{(s-3)}\Big]\frac{\partial f}{\partial a_3^{(s-2)}}.$$

The remaining $\sigma-2$ equations involve differential coefficients of f with regard to $a_1''', \ldots\ldots, a_1^{(\sigma)}$; $a_2'', \ldots\ldots, a_2^{(\sigma-1)}$; and $a_3', \ldots\ldots, a_3^{(\sigma-2)}$ only, and are therefore identically satisfied by *any* function of a_1, a_1', a_1'', a_2, a_2', a_3. There are altogether $\sigma+1$ equations, which form a complete system in $(\sigma+1)+\sigma+(\sigma-1)$ variables, and hence there are $2\sigma-1$ solutions. Writing only those terms of this system which involve differential coefficients with regard to a_1, a_1', a_1'', a_2 and a_3, we have

$$\begin{aligned} 0 &= \frac{\partial f}{\partial a_1} + 2a_1 \frac{\partial f}{\partial a_2} + 3a_2 \frac{\partial f}{\partial a_3}, \\ 0 &= \frac{\partial f}{\partial a_2} + 3a_1 \frac{\partial f}{\partial a_3} + \frac{\partial f}{\partial a_1'}, \\ 0 &= \frac{\partial f}{\partial a_3} + \frac{\partial f}{\partial a_1''}. \end{aligned}$$

These three equations in five variables form a complete system, with the two solutions,

$$I_2 = a_2 - a_1^2 - a_1' \quad \text{and} \quad I_3 = a_3 - 3a_2 a_1 + 2a_1^3 - a_1''.$$

I_3 is Cockle's cubicriticoid. Then

$$\frac{dI_2}{dx}, \frac{d^2 I_2}{dx^2}, \ldots, \frac{d^{\sigma-1} I_2}{dx^{\sigma-1}}; \frac{dI_3}{dx}, \frac{d^2 I_3}{dx^2}, \ldots, \frac{d^{\sigma-2} I_3}{dx^{\sigma-2}},$$

are the remaining solutions of the complete system, $2\sigma - 3$ in number. There are thus two, and only two, essential invariants involving only a_1, a_2, a_3 and their derivatives.

§8. Proceeding in the same manner, we find that there are three, and only three, essential invariants involving only $a_1, \ldots, a_4$ and their derivatives. They are the three solutions of the following complete system:

$$\begin{aligned} 0 &= \frac{\partial f}{\partial a_1} + 2a_1 \frac{\partial f}{\partial a_2} + 3a_2 \frac{\partial f}{\partial a_3} + 4a_3 \frac{\partial f}{\partial a_4}, \\ 0 &= \frac{\partial f}{\partial a_2} + 3a_1 \frac{\partial f}{\partial a_3} + 6a_2 \frac{\partial f}{\partial a_4} + \frac{\partial f}{\partial a_1'}, \\ 0 &= \frac{\partial f}{\partial a_3} + 4a_1 \frac{\partial f}{\partial a_4} + \frac{\partial f}{\partial a_1''}, \\ 0 &= \frac{\partial f}{\partial a_4} + \frac{\partial f}{\partial a_1'''}. \end{aligned}$$

I_2 and I_3 are solutions. Call the third solution I_4. Neglecting the terms in a_1' and a_1'' in the second and third equations respectively, we find that we have still a complete system of equations with *one* solution, which must be I_4, since it contains neither a_1' nor a_1''. We have therefore to solve the equations

$$\begin{aligned} 0 &= \frac{\partial f}{\partial a_1} + 2a_1\frac{\partial f}{\partial a_2} + 3a_2\frac{\partial f}{\partial a_3} + 4a_3\frac{\partial f}{\partial a_4}, \\ 0 &= \frac{\partial f}{\partial a_2} + 3a_1\frac{\partial f}{\partial a_3} + 6a_2\frac{\partial f}{\partial a_4}, \\ 0 &= \frac{\partial f}{\partial a_3} + 4a_1\frac{\partial f}{\partial a_4}, \\ 0 &= \frac{\partial f}{\partial a_4} + \frac{\partial f}{\partial a_1''}. \end{aligned}$$

The ordinary method of solving such a system gives as the one solution of this system,

$$I_4 = a_4 - 4a_3a_1 - 3a_2^2 + 12a_2a_1^2 - 6a_1^4 - a_1'''.$$

This is Cockle's quarticriticoid.

§9. By exactly similar reasoning we see that the "criticoid" of the k^{th} "degree" is the one solution of the following complete system:

$$\left.\begin{array}{r} \frac{\partial f}{\partial a_1} + 2a_1\frac{\partial f}{\partial a_2} + 3a_2\frac{\partial f}{\partial a_3} + \ldots\ldots + la_{l-1}\frac{\partial f}{\partial a_l} \qquad + \ldots\ldots + ka_{k-1}\frac{\partial f}{\partial a_k}, \\ \frac{\partial f}{\partial a_2} + 3a_1\frac{\partial f}{\partial a_3} + \ldots\ldots + \frac{l(l-1)}{2}a_{l-2}\frac{\partial f}{\partial a_l} \qquad + \ldots\ldots + \frac{k(k-1)}{2}a_{k-2}\frac{\partial f}{\partial a_k}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \ldots\ldots + \frac{l!}{(l-r)!\,r!}a_{l-r}\frac{\partial f}{\partial a_l} + \ldots\ldots + \frac{k!}{(k-r)!\,r!}a_{k-r}\frac{\partial f}{\partial a_k}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial f}{\partial a_{k-1}} + ka_1\frac{\partial f}{\partial a_k}, \\ \frac{\partial f}{\partial a_k} + \frac{\partial f}{\partial a_1^{(k-1)}}. \end{array}\right\} \quad (7)$$

From (4) we see that the r^{th} $(r = 1, 2, \ldots\ldots, k-1)$ equation of this system, being the coefficient of $\phi^{(r)}$ in the extended transformation, has the form

$$0 = \overset{k}{\underset{r}{\sum}}{}_{l}\,\frac{l!}{(l-r)!\;r!}\,a_{l-r}\frac{\partial f}{\partial a_l}, \qquad (a_0 = 1) \qquad (8)$$

The solution, I_k, of this system (7) will not be the same as the B_k given earlier, for this solution contains only $a_1, \ldots\ldots, a_k, a_1^{(k-1)}$, while B_k involves in addition to these all the derivatives $a_1', \ldots, a_1^{(k-2)}$. The two sets of invariants $B_2, \ldots, B_k$ and $I_2, \ldots, I_k$ are of course *functionally* equivalent.

If we assign to a_k the *weight* k and to $a_k^{(s)}$ the weight $(k+s)$, then I_2 is isobaric of weight 2, I_3 isobaric of weight 3, and I_4 isobaric of weight 4. We might therefore from analogy assume that I_k is an integral rational function, isobaric of weight k; thus:

$$I_k = a_k - a_1^{(k-1)} + c_{11}a_1a_{k-1} + (c_{12}a_1^2 + c_{22}a_2)\,a_{k-2}$$
$$+ (c_{13}a_1^3 + c_{23}a_2a_1 + c_{33}a_3)\,a_{k-3} + \ldots .$$

On substituting this value in the $(k-1)^{\text{th}}$ equation of the system (7) we find that $c_{11} = -k$. Then the $(k-2)^{\text{th}}$ equation gives $c_{12} = k(k-1)$, $c_{22} = -\frac{k(k-1)}{2}$, the $(k-3)^{\text{th}}$ equation will give c_{13}, c_{23}, c_{33}, and so on, the first equation finally giving the coefficient of a_1^k. In the series assumed for I_k no parenthesis is to contain a's of subscript greater than the subscript of the factor into which it is multiplied, otherwise terms will be repeated. We see thus that if I_k be assumed of this form, that values of c can be determined so that it is a solution of all of the equations. Therefore I_k is really isobaric of weight k, and its value may be found in this way by algebraic operations, that is, without integration.

§10. Having now the increments of $y^{(k)}$ and $a_k^{(s)}$ (see (3) and (6)), the work of finding the covariants is short. For the equation of the third order, we have from (3) and (6), (the factor δt being omitted):

$$\begin{aligned}
&\delta y = \phi y, \quad \delta y' = \phi' y + \phi y', \quad \delta y'' = \phi'' y + 2\phi' y' + \phi y'',\\
&\delta a_1 = -\phi', \quad \delta a_1' = -\phi'', \quad \delta a_1'' = -\phi''', \quad \delta a_2 = -(\phi'' + 2a_1\phi'),\\
&\delta a_2' = -(\phi''' + 2a_1\phi'' + 2a_1'\phi'), \quad \delta a_3 = -(\phi''' + 3a_1\phi'' + 3a_2\phi'),
\end{aligned}$$

Forming the extended transformation, and, as before, equating the coefficients of the different derivatives of ϕ to zero, we get

$$\begin{aligned}
0 &= y\frac{\partial f}{\partial y} + y'\frac{\partial f}{\partial y'} + y''\frac{\partial f}{\partial y''},\\
0 &= y\frac{\partial f}{\partial y'} + 2y'\frac{\partial f}{\partial y''} - \frac{\partial f}{\partial a_1} - 2a_1\frac{\partial f}{\partial a_2} - 2a_1'\frac{\partial f}{\partial a_2'} - 3a_2\frac{\partial f}{\partial a_3},\\
0 &= y\frac{\partial f}{\partial y''} - \frac{\partial f}{\partial a_1'} - \frac{\partial f}{\partial a_2} - 2a_1\frac{\partial f}{\partial a_2'} - 3a_1\frac{\partial f}{\partial a_3},\\
0 &= \frac{\partial f}{\partial a_1''} + \frac{\partial f}{\partial a_2'} + \frac{\partial f}{\partial a_3}.
\end{aligned}$$

This complete system of four equations in nine variables has five solutions, for the equations are evidently independent. Three of these solutions are I_2, I_3 and $\frac{dI_2}{dx}$, and are free from y. The remaining two involve y or its derivatives, and are absolute covariants. They are:

$$\frac{y' + a_1 y}{y} \quad \text{and} \quad \frac{y'' + 2a_1 y' + a_2 y}{y}.$$

These agree with the relative covariants given by Cockle, for since y is a relative covariant, the numerators of both of these absolute covariants must be relative covariants. Comparing Cockle's operative symbol, Δ, with the above equations, we see that it is equivalent to the second of them. For the *relative* invariants the zero in the left-hand member of the first of the above equations must be replaced by f (see §39). Cockle therefore had found only *one* of the four partial differential equations which his covariant satisfies.

§11. For the covariants of the general equation of the n^{th} order, we have, from (3) and (6), as the extended transformation,

$$\sum_{k=0}^{w}\sum_{m=0}^{k}\frac{k!}{(k-m)!\,m!}\phi^{(m)}y^{(k-m)}\frac{\partial f}{\partial y^{(k)}}$$
$$-\sum_{k=1}^{n}\sum_{s=0}^{w-k}\sum_{i=1}^{k}\sum_{m=i}^{s+i}\frac{s!\,k!\,\phi^{(m)}a_{k-i}^{(s+i-m)}}{(k-i)!\,i!\,(s+i-m)!\,(m-i)!}\frac{\partial f}{\partial a_k^{(s)}},$$

where $w \geqq n$ is an integer, and the transformation is so far extended that terms of weight not greater than w enter. On changing the order of summation, this becomes

$$\sum_{m=0}^{w}\phi^{(m)}\sum_{k=m}^{w}\frac{k!}{(k-m)!\,m!}y^{(k-m)}\frac{\partial f}{\partial y^{(k)}}$$
$$-\sum_{m=1}^{w}\phi^{(m)}\sum_{k=1}^{n}\sum_{s=\left\{\substack{m-k\\0}\right.}^{w-k}\sum_{i=\left\{\substack{m-s\\1}\right.}^{\left\{\substack{k\\m}\right.}\frac{s!\,k!\,a_{k-i}^{(s+i-m)}}{(k-i)!\,i!\,(s+i-m)!\,(m-i)!}\frac{\partial f}{\partial a_k^{(s)}}.$$

In this expression, when two upper limits are written, the smaller is chosen, and when two lower limits are written the larger one is to be used. Equating to zero the coefficient of $\phi^{(m)}$, we obtain the following complete system of partial differential equations:

$$\left.\begin{aligned}
&0=\sum_{0}^{w}{}_{k}\, y^{(k)}\,\frac{\partial f}{\partial y^{(k)}}\,,\\
&0=\sum_{m}^{w}{}_{k}\,\frac{k!}{(k-m)!\;m!}\,y^{(k-m)}\,\frac{\partial f}{\partial y^{(k)}}\\
&\qquad -\sum_{1}^{n}{}_{k}\;\sum_{\left\{\begin{smallmatrix}m-k\\0\end{smallmatrix}\right.}^{w-k}{}_{s}\;\sum_{\left\{\begin{smallmatrix}m-s\\0\end{smallmatrix}\right.}^{\left\{\begin{smallmatrix}k\\m\end{smallmatrix}\right.}{}_{i}\,\frac{s!}{(k-i)!\;i!}\,\frac{k!\,a_{k-i}^{(s+i-m)}}{(s+i-m)!\;(m-i)!}\,\frac{\partial f}{\partial a_k^{(s)}}\,.\\
&\qquad\qquad (m=1,\,2,\,\ldots\ldots,\,w)
\end{aligned}\right\}\quad(9)$$

These $w+1$ equations are linearly independent, for the first determinant of the matrix of the coefficients has the value $y^{w+1}\neq 0$. The number of variables is

$$w+1+\frac{w\,(w+1)-(w-n)(w-n+1)}{2}\,,$$

so that the number of solutions is $nw-\frac{1}{2}\,(n^2-n)$. Consider in particular the the case $w=n$. The number of absolute covariants is then $\frac{1}{2}\,n\,(n+1)$. Among this number we have the invariants $I_2,\ I_3,\ \ldots\ldots,\ I_n$, and their derivatives up to those of weight n. The total number of these solutions is:

$$(n-1)+(n-2)+\ldots\ldots+2+1=\tfrac{1}{2}\,n\,(n-1)\,.$$

There are consequently exactly $\frac{1}{2}\,n\,(n+1)-\frac{1}{2}\,n\,(n-1)=n$ other solutions, and these are

$$\frac{y'+a_1y}{y}\,,\quad\frac{y''+2a_1y'+a_2y}{y}\,,\ \ldots\ldots,\ \frac{y^{(n)}+na_1y^{(n-1)}+\ldots\ldots+a_ny}{y}\,.$$

We see that these quantities are solutions as follows: Let us use Cockle's notation

$$y_r=a_0y^{(r)}+ra_1y^{(r-1)}+\ldots\ldots+a_ry=\sum_{0}^{r}{}_{j}\,\frac{r!}{(r-j)!\;j!}\,a_j\,y^{(r-j)}\,;\qquad(a_0=1).$$

We have so determined $\delta y^{(k)}$ and $\delta a_k^{(s)}$ that $y_n=0$ is an invariant differential equation. But the expressions (3) and (6) for $\delta y^{(k)}$ and $\delta a_k^{(s)}$ do not involve n, the order of the equation, and would be the same if determined from any other equation $y_r = 0$. Therefore the equation $y_r = 0$, $(r = 1, 2, \ldots, n)$ must be an invariant differential equation for the transformation in question. That is,

$$y_r \equiv \rho v_r,$$

when the substitution $y = u . v$ is made. The factor ρ may be determined by considering the highest differential coefficient

$$y^{(r)} = uv^{(r)} + \ldots .$$

Hence

$$y_r = \rho v_r = uv_r,$$

$$\frac{y_r}{y} = \frac{v_r}{v},$$

and $\frac{y_r}{y}$ $(r = 1, 2, \ldots, n)$ are absolute covariants.

This same fact may also be easily proved by direct substitution in (9). When we substitute

$$y_r = \sum_0^r{}_j \frac{r!}{(r-j)!\, j!} a_j y^{(r-j)}$$

in any one of the last $w = n$ equations (9) we have only to retain in the second summation those terms which involve a_j; that is, we may write $s = 0$, $i = m$, so that the equations become

$$0 = \sum_m^n{}_k \frac{k!}{(k-m)!\ m!} y^{(k-m)} \frac{\partial f}{\partial y^k} - \sum_m^n{}_k \frac{k!}{(k-m)!\ m!} a_{k-m} \frac{\partial f}{\partial a_k} \equiv A_m f.$$
$$(m = 1, 2, \ldots, n)$$

We have

$$\frac{\partial y_r}{\partial y^{(k)}} = \frac{r!}{(r-k)!\ k!} a_{r-k}, \quad \frac{\partial y_r}{\partial a_k} = \frac{r!}{(r-k)!\ k!} y^{(r-k)}, \qquad (0 \leqq k \leqq r)$$

and

$$\frac{\partial y_r}{\partial y^{(k)}} = 0, \quad \frac{\partial y_r}{\partial a_k} = 0 \text{ when } k > r.$$

Hence

$$A_m y_r = \sum_m^r{}_k \frac{k!}{(k-m)!\ m!} \frac{r!}{(r-k)!\ k!} [y^{(k-m)} a_{r-k} - y^{(r-k)} a_{k-m}]$$
$$= \sum_m^r{}_k \frac{r!}{(k-m)!\ (r-k)!\ m!} y^{(k-m)} a_{r-k} - \sum_m^r{}_j \frac{r!}{(j-m)!\ (r-j)!\ m!} y^{(j-m)} a_{r-j}$$
$$= 0.$$

That is, y_r is a solution of the last n equations, and the first equation tells us that a solution must be homogeneous in the y's. Therefore y_r/y is a solution of the system (9). When $w > n$ the remaining solutions are obtained by differentiating those already found with regard to x.

Cockle had therefore in 1865 indicated the complete system of invariants and covariants of a homogeneous linear differential equation of the n^{th} order, for a transformation of the dependent variable of the form $y = v \,.\, u(x)$. *His* system contained more than the requisite number of solutions, for we have just seen that the solutions $\dfrac{d^p y_q}{dx^p}$ are not needed.

§12. Let us now take up the case in which the independent variable alone is transformed. Then

$$x_1 = \chi(x), \quad y_1 = y,$$

where x_1, y_1 are the transformed variables, and χ is an arbitrary function of x. These transformations form an infinite continuous group defined by differential equations. We get the infinitesimal transformation by writing $\chi = x + \xi(x)\,\delta t$. Then

$$\delta x = \xi(x)\,\delta t, \quad \delta y = 0,$$

and the symbol of the transformation is

$$\xi(x)\,\frac{\partial f}{\partial x}.$$

The following work is not made perfectly general, but is carried only far enough to get the covariant given by Cockle in 1875:

$$\begin{aligned}
\delta y' &= \frac{d\delta y}{dx} - y'\frac{d\delta x}{dx} = -\xi' y'\,\delta t. \\
\delta y^{(k)} &= \frac{d\delta y^{(k-1)}}{dx} - y^{(k)}\frac{d\delta x}{dx} = -y^{(k)}\xi'\,\delta t + \frac{d}{dx}\left(\delta y^{(k-1)}\right), \\
&= -\delta t\,.\left[y^{(k)}\xi' + \frac{d}{dx}\left(y^{(k-1)}\xi'\right) + \frac{d^2}{dx^2}\left(y^{(k-2)}\xi'\right) + \dots + \frac{d^{k-1}}{dx^{k-1}}\left(y'\xi'\right)\right] \\
&= -\delta t \sum_{0}^{k-1}{}_{s} \frac{d^s}{dx^s}\left(y^{(k-s)}\xi'\right) = -\delta t \sum_{0}^{k-1}{}_{s} \sum_{0}^{s}{}_{m} \frac{s\,!}{(s-m)!\;m\,!}\, y^{(k-m)}\xi^{(m+1)} \\
&= -\sum_{0}^{k-1}{}_{m}\, y^{(k-m)}\xi^{(m+1)} \sum_{m}^{k-1}{}_{s} \frac{s\,!}{(s-m)!\;m\,!}\,.
\end{aligned}$$

Since $\sum_{m}^{k-1}{}_{s} \frac{s\,!}{(s-m)\,!\; m\,!} = \frac{k\,!}{(k-m-1)\,!\,(m+1)\,!}$, this becomes

$$\delta y^{(k)} = -\,\delta t \sum_{0}^{k-1}{}_{m} \frac{k\,!}{(k-m-1)\,!\;(m+1)\,!}\, y^{(k-m)}\xi^{(m+1)}. \tag{10}$$

As in the case for change of the dependent variable, we have

$$\delta y^{(n)} + n a_1\, \delta y^{(n-1)} + \ldots\ldots + a_n\, \delta y + n y^{(n-1)}\, \delta a_1 + \ldots\ldots + y\, \delta a_n = 0.$$

On substituting the values of $\delta y^{(n)}$, $\delta y^{(n-1)}$, $\delta y^{(n-2)}$ as given by (10), this becomes

$$0 = -n\xi' y^{(n)} - \begin{array}{r|r|l} \frac{n(n-1)}{2}\,\xi'' & y^{(n-1} - \frac{n(n-1)(n-2)}{3\,!}\,\xi''' & y^{(n-2)} + \ldots\ldots \\ -\,n(n-1)\,a_1\xi' & -\,\frac{n(n-1)(n-2)}{2}\,a_1\xi'' & \\ +\,n\delta a_1 & -\,\frac{n(n-1)(n-2)}{2}\,a_2\xi' & \\ & +\,\frac{n(n-1)}{2}\,\delta a_2 & \end{array}$$

Eliminate the term involving $y^{(n)}$ by means of (2) and equate the coefficients of $y^{(n-1)}$ and $y^{(n-2)}$ to zero. This gives

$$\delta a_1 = \left(\frac{n-1}{2}\,\xi'' - a_1\xi'\right)\delta t,$$

$$\delta a_2 = \left(\frac{n-2}{3}\,\xi''' + (n-2)\,a_1\xi'' - 2a_2\,\xi'\right)\delta t,$$

$$\delta a_1' = \delta\,\frac{da_1}{dx} = \frac{d}{dx}\,\delta a_1 - a_1'\xi'\,\delta t = \left(\frac{n-1}{2}\,\xi''' - a_1\xi'' - 2a_1'\xi'\right)\delta t.$$

The extended transformation is then

$$\xi\,\frac{\partial f}{\partial x} - \xi' y'\,\frac{\partial f}{\partial y'} + \left(\frac{n-1}{2}\,\xi'' - a_1\xi'\right)\frac{\partial f}{\partial a_1}$$
$$+ \left(\frac{n-2}{3}\,\xi''' + (n-2)\,a_1\xi'' - 2a_2\xi'\right)\frac{\partial f}{\partial a_2}$$
$$+ \left(\frac{n-1}{2}\,\xi''' - a_1\,\xi'' - 2a_1'\xi'\right)\frac{\partial f}{\partial a_1'}\,.$$

Equating the coefficients of the different derivatives of ξ to zero, we have

$$\begin{aligned} 0 &= y' \frac{\partial f}{\partial y'} + a_1 \frac{\partial f}{\partial a_1} + 2a_2 \frac{\partial f}{\partial a_2} + 2a_1' \frac{\partial f}{\partial a_1'}, \\ 0 &= \frac{n-1}{2} \frac{\partial f}{\partial a_1} + (n-2)\, a_1 \frac{\partial f}{\partial a_2} - a_1 \frac{\partial f}{\partial a_1'}, \\ 0 &= \frac{n-2}{3} \frac{\partial f}{\partial a_2} + \frac{n-1}{2} \frac{\partial f}{\partial a_1'}. \end{aligned}$$

By the ordinary methods the one solution of this complete system of three equations in four variables is found to be

$$\left[a_1' + \frac{3n-1}{2n-2} a_1^2 - \frac{3(n-1)}{2(n-2)} a_2\right] \frac{1}{y'^2}, \qquad n > 2.$$

This is Cockle's "differential quadricriticoid." Other invariants found by extending the group further are not here given because they are simply the solutions of equations to be given later. We have now found all of Cockle's invariants by Lie's methods, and proceed to the general case.

CHAPTER 3.

Invariants of the General Homogeneous Linear Differential Equation in Two Variables, for Transformation of Both Variables.

§13. It is known* that the most general point transformation which transforms the equation

$$a_0 y^{(n)} + n a_1 y^{(n-1)} + \ldots + a_n y = \sum_0^n \frac{n!}{(n-s)!\, s!} a_s y^{(n-s)} = 0, \quad (a_0 = 1) \qquad (2)$$

into an equation of the same form is, when $n > 1$, one of the form

$$x_1 = \chi(x), \quad y_1 = y \Psi(x). \qquad (11)$$

These transformations (11) form an infinite continuous group defined by differential equations. Lie's theory may therefore be used in finding all the invariants of the equation (2) for the transformations (11).

§14. Let us consider the effect of a finite transformation before taking up the infinitesimal. The following treatment follows closely that for finding the factor by which a relative differential invariant in $x, y, y', y'', \ldots$ is multiplied when subjected to a point transformation. It is thought best to make this treatment only general enough for the purposes of this paper, and not to give the factor in determinant form, as might be done. We have

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* P. Stäckel, Journal für Math., CXI, pp. 290–302.

$$
\begin{aligned}
y_1 &= y\Psi(x), \quad x_1 = \chi(x), \\
y_1' &= \frac{\Psi}{\chi'}\, y' + \frac{\Psi'}{\chi'}\, y, \\
y_1'' &= \frac{\Psi}{\chi'^2}\, y'' + \frac{2\Psi'\chi' - \Psi\chi''}{\chi'^3}\, y' + \frac{\chi'\Psi'' - \chi''\Psi'}{\chi'^3}\, y, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
y_1^{(\mu)} &= \frac{\Psi}{\chi'^{\mu}}\, y^{(\mu)} + \sum_{s=0}^{\mu-1} \beta_{\mu s}(x) \,.\, y^{(s)}.
\end{aligned}
\tag{12}
$$

When these values are substituted in

$$y_1^{(n)} + nb_1 y_1^{(n-1)} + \ldots + b_n y_1 = 0$$

we must get a multiple of (2), for (2) must be an invariant equation, and the substitution (12) is linear in the $y^{(k)}$'s; therefore

$$y_1^{(n)} + nb_1 y_1^{(n-1)} + \ldots + b_n y_1 \equiv \frac{\Psi}{\chi'^n}\left[y^{(n)} + na_1 y^{(n-1)} + \ldots + a_n y\right],$$

$$\sum_{s=0}^{n} \frac{n!}{(n-s)!\, s!}\, b_s y_1^{(n-s)} \equiv \frac{\Psi}{\chi'^n} \sum_{s=0}^{n} \frac{n!}{(n-s)!\, s!}\, a_s y^{(n-s)}. \tag{13}$$

Substitution of (12) in (13) and comparison of the coefficients of $y^{(n-j)}$ gives

$$a_j = \chi'^j b_j + \frac{\chi'^n}{\Psi} \sum_{k=0}^{j-1} \frac{(n-j)!\, j!}{(n-k)!\, k!}\, b_k \beta_{n-j,\, n-k}. \quad (j = 1, 2, \ldots, n). \tag{14}$$

Solving these n equations for b_j $(j = 1, 2, \ldots, n)$ gives

$$b_j = \frac{1}{\chi'^j}\, a_j + \sum_{s=0}^{j-1} \gamma_{js}(x) \,.\, a_s, \quad (j = 1, 2, \ldots, n) \tag{15}$$

Differentiating (15) k times with regard to x gives

$$b_j^{(k)} = \frac{1}{\chi'^{j+k}}\, a_j^{(k)} + \sum_{s=0}^{j} \sum_{i=0}^{k} \vartheta_{jksi}(x) \,.\, a_s^{(i)}, \text{ where } \vartheta_{jkjk} = 0. \tag{16}$$

$$(j = 1, 2, \ldots, n, \; k = 0, 1, 2, \ldots)$$

The formula (16) includes (15), and (16) and (12) together with (11) are the finite equations of the extended group. Since (12) and (16) are linear substitutions, any integral rational function in $y^{(\mu)}$ and $a_j^{(k)}$ will be transformed into an integral rational function whose degree taken in either set of variables alone, or in both together, will be the same as that of the original function.

§15. Now make the transformation

$$x_1 = x, \quad y_1 = Cy,$$

which is included in (11). Then

$$y_1^{(\mu)} = Cy^{(\mu)}, \quad \text{and} \quad b_j^{(k)} = a_j^{(k)}.$$

Therefore any differential invariant

$$I(x, y, y' \dots y^{(\nu)}; \quad a_1 \dots a_n, \dots a_j^{(k)} \dots)$$

must be homogeneous in $y, y', \dots, y^{(\nu)}$ of degree zero, and any invariant differential equation must be homogeneous in $y, y', \dots, y^{(\nu)}$

Next make the transformation,

$$y_1 = y, \quad x_1 = Cx,$$

which is also included in (11). Then

$$y_1^{(\mu)} = C^{-\mu} y^{(\mu)}, \quad b_j^{(k)} = C^{-(j+k)} a_j^{(k)}.$$

Let us assign to $[y^{(\mu)}]^{\nu}$ the weight $\mu\nu$, and to $[a_j^{(k)}]^{l}$ the weight $(j+k)l$. Then these equations show that any invariant must be isobaric in the $y^{(\mu)}$'s and $a_j^{(k)}$'s of weight zero, and that any invariant differential equation must be isobaric in the $y^{(\mu)}$'s and $a_j^{(k)}$'s. Hence we have the theorem:

Theorem I.

Any absolute invariant of the linear differential equation (2) for the group of transformations (11) must be homogeneous in the $y^{(\mu)}$'s of degree zero and isobaric in the $y^{(\mu)}$'s and $a_j^{(k)}$'s of weight zero. An invariant equation (or relative invariant) must be homogeneous in the $y^{(\mu)}$'s and isobaric in the $y^{(\mu)}$'s and $a_j^{(k)}$'s.

This theorem must hold for every subgroup of (11) which contains the two transformations

$$\begin{cases} y_1 = Cy, \\ x_1 = x, \end{cases} \quad \begin{cases} y_1 = y, \\ x_1 = Cx. \end{cases}$$

§16. Let us consider an integral rational function which is a relative invariant. Then by Theorem I it is homogeneous in the $y^{(\mu)}$'s, say of degree λ, and isobaric in the $y^{(\mu)}$'s and $a_j^{(k)}$'s, say of weight w. Represent this by

$$\Omega^{(\lambda,\, w)} = 0 .$$

This equation is invariant. $\Omega^{(\lambda,\, w)}$ must be composed of individual terms, each of which is of degree λ in the $y^{(\mu)}$'s and of weight w in the $y^{(\mu)}$'s and $a_j^{(k)}$'s. The formulæ (12) and (16) show that any such term is to be replaced by the same term multiplied by Ψ^λ/χ'^w, plus a function of weight *less* than w. Then

$$\Omega_1^{(\lambda,\, w)} \equiv \frac{\psi^\lambda}{\chi'^w}\, \Omega^{(\lambda,\, w)} + \Theta,$$

where Θ is an integral rational function, none of whose terms is of weight as great as w. The left-hand member of this identity vanishes when we place $\Omega^{(\lambda,\, w)} = 0$, for this is an invariant equation. Hence Θ is identically zero, for a function of weight less than w cannot vanish by means of one of weight w. That is,

$$\Omega_1^{(\lambda,\, w)} \equiv \frac{\Psi^\lambda}{\chi'^w}\, \Omega^{(\lambda,\, w)}. \tag{17}$$

This formula (17) shows how an invariant integral rational equation is transformed. The same is true if $\Omega^{(\lambda,\, w)}$ is any root of an integral rational function, extensions of the definition of degree and weight to such a function being made. These are the only functions which arise in the present article. If an invariant irrational algebraic equation were given, it could be rationalized, and then (17) would give the factor by which the invariant is multiplied when a transformation (11) is made.

Theorem II.—Whenever three relative invariants are known, an absolute invariant may be constructed by algebraic processes.

Let $\Omega^{(\lambda_1,\, w_1)}$, $\Omega^{(\lambda_2,\, w_2)}$ $\Omega^{(\lambda_3,\, w_3)}$ be the three relative invariants. Then (17) shows that

$$[\Omega^{(\lambda_1,\, w_1)}]^{\mu_1}\, [\Omega^{(\lambda_2,\, w_2)}]^{\mu_2}\, [\Omega^{(\lambda_3,\, w_3)}]^{\mu_3} \tag{18}$$

is an absolute invariant, if μ_1, μ_2, μ_3 be determined from the equations

$$\begin{aligned} \lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 &= 0, \\ w_1\mu_1 + w_2\mu_2 + w_3\mu_3 &= 0, \end{aligned}$$

that is, if

$$\mu_1 : \mu_2 : \mu_3 = \begin{vmatrix} \lambda_2 & \lambda_3 \\ w_2 & w_3 \end{vmatrix} : \begin{vmatrix} \lambda_3 & \lambda_1 \\ w_3 & w_1 \end{vmatrix} : \begin{vmatrix} \lambda_1 & \lambda_2 \\ w_1 & w_2 \end{vmatrix}. \tag{19}$$

This formula always gives possible values of $\mu_1:\mu_2:\mu_3$, except when

$$\lambda_1:\lambda_2:\lambda_3 = w_1:w_2:w_3,$$

in which case it is necessary to know only two different invariant equations. This case occurs when two of the given relative invariants are free from y and its derivatives; it is then not necessary to know a third. No one of the Ω's may be an absolute invariant if (18) is to give a *new* absolute invariant.

This theorem is the exact analogon of a well-known theorem of Lie concerning invariant differential equations of curves in a plane.

Theorem III.—If an absolute invariant be a rational algebraic function, both its numerator and denominator are relative invariants, of the same degree and weight. Let $I = U/V$ be an absolute invariant, where U and V are integral rational functions, which by (12) and (16) transform into integral rational functions, say U_1 and V_1 respectively.

$$\frac{U}{V} = k$$

is an invariant equation, and this invariance is not destroyed if we clear of fractions.

$$U - kV = 0$$

is therefore an invariant integral rational equation, and must be homogeneous in the y's and isobaric (Theorem I). Say the degree in the y's is λ and the weight w. Then by (17)

$$U_1 - kV_1 = \frac{\Psi^\lambda}{\chi^{w}}(U - kV).$$

This equation is true for all values of k. Hence

$$U_1 = \frac{\Psi^\lambda}{\chi^{w}}\,U, \quad V_1 = \frac{\Psi^\lambda}{\chi^{w}}\,V. \qquad \text{Q. E. D.}$$

Theorem IV.—The total derivative with regard to x of an absolute invariant is a relative invariant of next higher order.

If I be an absolute invariant, then

$$I_1 = I,$$

and this is an identity when we make use of (11), (12) and (16). Hence we may differentiate totally with regard to x. This gives

$$\frac{dI_1}{dx_1}=\frac{1}{\chi'}\,\frac{dI}{dx},$$

which shows that $\frac{dI}{dx}$ is a relative invariant, homogeneous of degree zero in the y's and isobaric of weight unity. Its order in both y and the a's is evidently one greater than that of I.

As a corollary, we see that if we have two absolute invariants, I_1 and I_2, that then

$$\frac{dI_2}{dx}\Big/\frac{dI_1}{dx}, \quad \text{or} \quad \frac{dI_2}{dI_1}$$

is an absolute invariant, and

$$\frac{dI_2}{dI_1},\quad \frac{d^2I_2}{dI_1^2},\quad \frac{d^3I_2}{dI_1^3},\;\ldots. \tag{20}$$

is an infinite series of absolute invariants in ascending order. For transformation of the dependent variable alone, we may write $I_1\equiv x$. This justifies the method used at the end of §6.

§17. The preceding theorems enable us to derive an infinite series of absolute invariants in ascending order from *two known relative invariants*. We always know a third relative invariant, namely, y, for we have $y_1=\Psi y$. Then applying Theorem II gives an absolute invariant, and IV gives a new relative invariant of next higher order. Using II again yields a second absolute invariant, and this alternating process may be continued, or (20) may be used, and we thus get an infinite series of absolute invariants in ascending order.

§18. If a *single absolute invariant*, say

$$I=\frac{U^{(\lambda,\,w)}}{V^{(\lambda,\,w)}},$$

be known, Theorems II, III, IV, together with the foregoing remarks, show that

$$J=\left[\frac{y^{\lambda}}{V^{(\lambda,\,w)}}\right]^{\frac{1}{w}}\frac{dI}{dx} \tag{21}$$

is an absolute invariant of order one greater than that of I. A repetition of the process gives an infinite series of absolute invariants from a single absolute invariant.

§19. For subgroups of the general group (11) it is often possible to find an infinite series of absolute invariants by differentiation processes from a *single* relative invariant. Thus if

$$\Psi = C\chi'^{\nu},$$

where C is an arbitrary and ν a known constant, we have a subgroup of (11). Suppose that for this subgroup we know a relative invariant U of degree λ in the y's and isobaric of weight w. Then we have

$$U_1 = \frac{\Psi^{\lambda}}{\chi'^{w}}\, U = C^{\lambda}\chi'^{\lambda\nu - w}U;\; y_1 = C\chi'^{\nu}y;$$

$$U_1' = C^{\lambda}\chi'^{\lambda\nu - w - 1}U' + C^{\lambda}(\lambda\nu - w)\chi'^{\lambda\nu - w - 2}\chi''U;\; y_1' = C\chi'^{\nu - 1}y' + C\nu\chi'^{\nu - 2}\chi''y.$$

The elimination of χ', χ'', and C between these four equations shows that

$$\frac{\nu U'y - (\lambda\nu - w)\,Uy'}{U^{\frac{w+1}{w}}\,y^{\frac{w-\lambda}{w}}} \tag{22}$$

is an absolute invariant for the subgroup

$$x_1 = \chi(x), \quad y_1 = Cy\chi'^{\nu},$$

when U is a relative invariant, of degree λ in the y's, and isobaric of weight w.

For other subgroups other methods of deriving new invariants from known invariants by differentiation can be developed (cf. Forsyth's quadriderivative and Jacobian processes)*, but as we have now all the methods we need, we shall not go into this. (Cf. §51.)

§20. With these general remarks on the methods of finding new invariants from known invariants, we return to the actual process of finding the invariants by the method of infinitesimal transformations.

The symbol of the infinitesimal transformations of the group (11) is

$$Xf \equiv \xi(x)\frac{\partial f}{\partial x} + \phi(x)\,y\,\frac{\partial f}{\partial y}, \tag{23}$$

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* Phil. Trans., I, 1888, pp. 407–418.

where ξ and ϕ are arbitrary functions of x alone. This transformation must be extended by taking into account the variation of $y^{(k)}$ and $a_j^{(k)}$. We have

$$\begin{aligned}\delta y^{(k)} &= \delta \frac{dy^{(k-1)}}{dx} = -y^{(k)}\xi'\,\delta t + \frac{d}{dx}(\delta y^{(k-1)})\\ &= -\left[y^{(k)}\xi' + \frac{d}{dx}(y^{(k-1)}\xi') + \dots + \frac{d^{k-1}}{dx^{k-1}}(y'\xi')\right]\delta t + \frac{d^k}{dx^k}(\delta y)\\ &= \left[-\sum_{s=0}^{k-1}\frac{d^s}{dx^s}(y^{(k-s)}\xi') + \frac{d^k}{dx^k}(\phi y)\right]\delta t.\end{aligned}$$

Making use of Euler's theorem for the differentiation of a product, and of the formula

$$\sum_{s=m}^{k-1}\frac{s!}{(s-m)!\,m!} = \frac{k!}{(k-m-1)!\,(m+1)!},$$

we find, in the same way that (10) was found,

$$\delta y^{(k)} = \sum_{m=0}^{k}\frac{k!}{(k-m)!\,(m+1)!}\left[(m+1)\,\phi^{(m)} - (k-m)\,\xi^{(m+)}\right]y^{(k-m)}\,\delta t. \tag{24}$$

§21. The variation of (2) gives

$$\sum_{s=1}^{n}\frac{n!}{(n-s)!\,s!}\,y^{(n-s)}\,\delta a_s + \sum_{s=0}^{n}\frac{n!}{(n-s)!\,s!}\,a_s\,\delta y^{(n-s)} = 0.$$

On substituting in this equation the value of $\delta y^{(n-s)}$ from (24) we get

$$\begin{aligned}&\sum_{s=1}^{n}\frac{n!}{(n-s)!\,s!}\,y^{(n-s)}\delta a_s\\ &\quad = -\sum_{s=0}^{n}\frac{n!\,a_s}{(n-s)!\,s!}\sum_{m=0}^{n-s}\frac{(n-s)!\,y^{(n-m-s)}}{(n-m-s)!\,(m+1)!}\left[(m+1)\,\phi^{(m)} - (n-m-s)\,\xi^{(m+1)}\right]\delta t\\ &\quad = -\sum_{s=0}^{n}\sum_{j=s}^{n}\frac{n!\,a_s y^{(n-j)}\left[(j-s+1)\,\phi^{(j-s)} - (n-j)\,\xi^{(j-s+1)}\right]}{s!\,(n-j)!\,(j-s+1)!}\,\delta t\\ &\quad = -\sum_{j=0}^{n}\sum_{s=0}^{j}\frac{n!\,a_s y^{(n-j)}\left[(j-s+1)\,\phi^{(j-s)} - (n-j)\,\xi^{(j-s+1)}\right]}{s!\,(n-j)!\,(j-s+1)!}\,\delta t\\ &\quad = -\sum_{j=1}^{n}\frac{n!\,y^{(n-j)}}{(n-j)!\,j!}\sum_{s=0}^{j-1}\frac{j!\,a_s}{s!\,(j-s+1)!}\left[(j-s+1)\,\phi^{(j-s)} - (n-j)\,\xi^{(j-s+1)}\right]\delta t\\ &\qquad -\sum_{j=0}^{n}\frac{n!}{j!\,(n-j)!}\,a_j y^{(n-j)}\left[\phi - (n-j)\,\xi'\right]\delta t.\end{aligned}$$

But in consequence of (2), $\sum_0^n{}_j \frac{n!}{j!\,(n-j)!}\, a_j\, y^{(n-j)}(\phi - n\xi') = 0$. Hence we finally get by equating coefficients of $y^{(n-j)}$,

$$\delta a_j = \left\{ -j a_j \xi' - \sum_0^{j-1}{}_s \frac{j!\ a_s}{(j-s+1)!\ s!} \left[(j-s+1)\,\phi^{(j-s)} - (n-j)\,\xi^{(j-s+1)} \right] \right\} \delta t. \quad (25)$$

$$(j = 1, 2, \ldots, n)$$

§22. In extending the group of transformations we also require $\delta a_j^{(k)}$.

$$\delta a_j^{(k)} = \delta \frac{d a_j^{(k-1)}}{dx} = -\xi' a_j^{(k)}\, \delta t + \frac{d}{dx}\left(\delta a_j^{(k-1)}\right).$$

$$= -\left[a_j^{(k)} \xi' + \frac{d}{dx}\left(a_j^{(k-1)} \xi'\right) + \ldots + \frac{d^{k-1}}{dx^{k-1}}\left(a_j' \xi'\right) \right] \delta t + \frac{d^k}{dx^k}\left(\delta a_j\right).$$

The quantity within the bracket is the same as that encountered in finding $\delta y^{(k)}$, a_j replacing y. Hence

$$\delta a_j^{(k)} = -\sum_0^k{}_m \frac{k!\ \xi^{(m+1)}}{(k-m-1)!\ (m+1)!}\, a_j^{(k-m)}\, \delta t + \frac{d^k}{dx^k}\left(\delta a_j\right)$$

$$= -\sum_0^k{}_m \frac{k!\ (k-m)}{(k-m)!\ (m+1)!}\, \xi^{(m+1)} a_j^{(k-m)} \delta t - j\,\delta t \sum_0^k{}_m \frac{k!}{(k-m)!\ m!}\, \xi^{(m+1)} a_j^{(k-m)}$$

$$- \delta t \,.\, \sum_0^{j-1}{}_s \frac{j!}{(j-s+1)!\ s!}\ \frac{d^k}{dx^k} \left\{ \left[(j-s+1)\,\phi^{(j-s)} - (n-j)\,\xi^{(j-s+1)} \right] a_s \right\}.$$

Combining the first two summations into one, and changing the letter of summation, gives

$$\delta a_j^{(k)} = -\,\delta t \left[\begin{array}{l} \displaystyle\sum_0^k{}_s \frac{k!\ [j(k+1) - s(j-1)]}{s!\ (k-s+1)!}\, a_j^{(s)} \xi^{(k-s+1)} \\ \displaystyle + \sum_0^{j-1}{}_s \frac{j!}{s!\ (j-s+1)!}\ \frac{d^k}{dx^k} \left\{ \left[(j-s+1)\,\phi^{(j-s)} - (n-j)\,\xi^{(j-s+1)} \right] a_s \right\} \end{array} \right]. \quad (26)$$

The form (26) is useful for some purposes, but in order to find the equations which give the invariants, the indicated differentiation under the second summation sign must actually be carried out, and then the summations arranged according to $\xi^{(s)}$ and $\phi^{(s)}$. Performing the differentiation gives

$$\begin{aligned}\frac{\delta a_j^{(k)}}{\delta t} = &-\mathop{\sum\nolimits_s}\limits_{1}^{k+1} \frac{k!\; a_j^{(k-s+1)}[k+1+s(j-1)]}{(k-s+1)!\; s!}\,\xi^{(s)} \\ &-\mathop{\sum\nolimits_s}\limits_{0}^{j-1}\;\mathop{\sum\nolimits_m}\limits_{j-s}^{k+j-s} \frac{j!\; k!\; a_s^{(k+j-s-m)}}{(j-s)!\; s!\; (k+j-s-m)!\; (s-j+m)!}\,\phi^{(m)} \\ &+\mathop{\sum\nolimits_s}\limits_{0}^{j-1}\;\mathop{\sum\nolimits_m}\limits_{j+1-s}^{k+j+1-s} \frac{j!\; k!\; (n-j)\, a_s^{(k+j+1-s-m)}}{(j-s+1)!\; s!\; (k+j-s-m+1)!\; (m+s-j-1)!}\,\xi^{(m)}.\end{aligned}$$

In the double sums the order of summation must now be changed so that the summation with regard to m comes first.

§23. In changing the order of summation the typical term will remain the same, only the *limits* of summation being changed. The limits of summation are changed as in the following example:

$\mathop{\sum\nolimits_s}\limits_{0}^{j-1}\;\mathop{\sum\nolimits_m}\limits_{j-s}^{k+j-s}$ shows that $j-s \leqq m \leqq k+j-s$. Hence if m be first chosen, we have for s,

$$j-m \leqq s \leqq k+j-m.$$

s also lies between other limits, namely: $0 \leqq s \leqq j-1$, and sometimes one set and sometimes another must be chosen. The third inequality combined with the first gives

$$1 < m \leqq k+j,$$

$$\therefore \mathop{\sum\nolimits_s}\limits_{0}^{j-1}\;\mathop{\sum\nolimits_m}\limits_{j-s}^{k+j-s} = \mathop{\sum\nolimits_m}\limits_{1}^{k+j}\;\mathop{\sum\nolimits_s}\limits_{\text{larger of}\left\{\begin{smallmatrix} j-m \\ 0 \end{smallmatrix}\right.}^{\text{smaller of}\left\{\begin{smallmatrix} k+j-m \\ j-1 \end{smallmatrix}\right.}.$$

In all of the following work if two (or more) upper limits are written the smaller is to be chosen, and if two (or more) lower limits are written the larger is to be chosen.

§24. Changing the orders of summation in this way gives

$$
\begin{aligned}
\frac{\delta a_j^{(k)}}{\delta t} = &-\sum_{s=1}^{k+1}\frac{k!\,[k+1+s(j-1)]}{(k-s+1)!\,s!}\,a^{(k-s+1)}\xi^{(s)} \\
&-\sum_{s=1}^{k+j}\phi^{(s)}\sum_{m=\left\{\begin{smallmatrix}0\\ j-s\end{smallmatrix}\right.}^{\left\{\begin{smallmatrix}k+j-s\\ j-1\end{smallmatrix}\right.}\frac{j!\,k!\,a_m^{(k+j-s-m)}}{(j-m)!\,m!\,(k+j-s-m)!\,(s+m-j)!} \\
&+\sum_{s=2}^{k+j+1}\xi^{(s)}\sum_{m=\left\{\begin{smallmatrix}0\\ j+1-s\end{smallmatrix}\right.}^{\left\{\begin{smallmatrix}j-1\\ k+j+1-s\end{smallmatrix}\right.}\frac{j!\,k!\,(n-j)\,a_m^{(k+j+1-s-m)}}{(j+1-m)!\,m!\,(k+j+1-s-m)!\,(m+s-j-1)!}. \qquad (27)\\
&(j=1,\,2,\,\ldots,\,n;\ k=0,\,1,\,2,\,\ldots;\ a_0=1)
\end{aligned}
$$

Formula (27) includes (25) as a special case, so that we need use only (27) and (24) in forming the extended group.

§25. It has been already stated that $a_j^{(k)}$ has the weight $(k+j)$, and that $y^{(\mu)}$ has the weight μ. If then we consider only those extended transformations which involve no $y^{(\mu)}$ of weight greater than w_1 and no $a_j^{(k)}$ of weight greater than w_2, we have as the required extended transformation

$$
X^{(w_1 w_2)}f = Xf + \sum_{k=1}^{w_1}\frac{\delta y^{(k)}}{\delta t}\cdot\frac{\partial f}{\partial y^{(k)}} + \sum_{j=1}^{\left\{\begin{smallmatrix}w_2\\ n\end{smallmatrix}\right.}\sum_{k=0}^{w_2-j}\frac{\delta a_j^{(k)}}{\delta t}\cdot\frac{\partial f}{\partial a_j^{(k)}}.
$$

In this formula the values of $\frac{\delta y^{(k)}}{\delta t}$ from (24) and of $\frac{\delta a_j^{(k)}}{\delta t}$ from (27) are to be substituted. The result will involve double, triple and quadruple summations. The order of summation must then be so changed that the summations with regard to $\xi^{(s)}$ and $\phi^{(s)}$ come first. When this is done, we get *the extended transformation*:

$$
\begin{aligned}
X^{(w_1 w_2)}f = {}&\xi\frac{\partial f}{\partial x} + \sum_{s=0}^{w_1}\phi^{(s)}\sum_{k=s}^{w_1}\frac{k!\,y^{(k-s)}}{(k-s)!\,s!}\,\frac{\partial f}{\partial y^{(k)}} - \sum_{s=1}^{w_1}\xi^{(s)}\sum_{k=s}^{w_1}\frac{k!\,y^{(k-s+1)}}{(k-s)!\,s!}\cdot\frac{\partial f}{\partial y^{(k)}} \\
&-\sum_{s=1}^{w_2}\phi^{(s)}\sum_{j=1}^{\left\{\begin{smallmatrix}w_2\\ n\end{smallmatrix}\right.}\sum_{k=\left\{\begin{smallmatrix}s-j\\ 0\end{smallmatrix}\right.}^{w_2-j}\sum_{m=\left\{\begin{smallmatrix}0\\ j-s\end{smallmatrix}\right.}^{\left\{\begin{smallmatrix}j-1\\ k+j-s\end{smallmatrix}\right.}\frac{j!\,k!\,a_m^{(k+j-s-m)}}{(j-m)!\,m!\,(k+j-s-m)!\,(s+m-j)!}\,\frac{\partial f}{\partial a_j^{(k)}} \\
&-\sum_{s=1}^{w_2}\xi^{(s)}\sum_{j=1}^{\left\{\begin{smallmatrix}n\\ w_2-s+1\end{smallmatrix}\right.}\sum_{k=s-1}^{w_2-j}\frac{k!\,[k+1+s(j-1)]}{(k+1-s)!\,s!}\,a_j^{(k+1-s)}\frac{\partial f}{\partial a_j^{(k)}} \\
&+\sum_{s=2}^{w_2+1}\xi^{(s)}\sum_{j=1}^{\left\{\begin{smallmatrix}w_2\\ n-1\end{smallmatrix}\right.}\sum_{k=\left\{\begin{smallmatrix}0\\ s-j-1\end{smallmatrix}\right.}^{w_2-j}\sum_{m=\left\{\begin{smallmatrix}j+1-s\\ 0\end{smallmatrix}\right.}^{\left\{\begin{smallmatrix}j-1\\ k+j+1-s\end{smallmatrix}\right.} \\
&\qquad\frac{j!\,k!\,(n-j)\,a_m^{(k+j+1-s-m)}}{(j+1-m)!\,m!\,(k+j+1-s-m)!\,(m+s-j-1)!}\,\frac{\partial f}{\partial a_j^{(k)}}. \qquad (28)
\end{aligned}
$$

§26. If $F(y^{(\mu)}, a_j^{(k)})$ is to be invariant for this transformation, then $X^{(w_1, w_2)}F = 0$. This equation breaks up into a number of equations, obtained by equating the coefficients of $\phi^{(s)}$ and $\xi^{(s)}$ to zero, for all possible values of s. The general case will not be here considered, but only the two special cases, $w_1 = 0,\ w_2 = w$, and $w_1 = w_2 = w$. The first of these will yield what are ordinarily known as invariants, that is, the invariant functions so obtained will not involve y or its derivatives; the second yields, in addition to those of the first class, the so-called covariants. In writing these equations those coming from $\phi^{(s)}$ are always given first, and then those from $\xi^{(s)}$. The former class of equations yield the invariants for transformations of y alone, the latter the invariants for transformations of x alone, while the common solutions of both sets of equations are the invariants for change of both x and y. Either set of equations alone forms a *complete* system. We have then the following complete system of

Equations to be satisfied by invariants involving only $a_j^{(k)}$, where $k + j \leq w$, for transformation of both x and y:

$$\left.\begin{aligned}
0 = -B_s f \;&\equiv \sum_{1}^{\left\{\substack{w\\ n}\right.}\!{}_j \sum_{\left\{\substack{s-j\\ 0}\right.}^{w-j}\!{}_k \sum_{\left\{\substack{0\\ j-s}\right.}^{\left\{\substack{j-1\\ k+j-s}\right.}\!{}_m \frac{j!\,k!\;a_m^{(k+j-s-m)}}{(j-m)!\,m!\,(k+j-s-m)!\,(m+s-j)!}\,\frac{\partial f}{\partial a_j^{(k)}}, \\
&\qquad (s = 1,\ 2, \ldots, w) \\
0 = -A_1 f \;&\equiv \sum_{1}^{\left\{\substack{w\\ n}\right.}\!{}_j \sum_{0}^{w-j}\!{}_k\,(k+j)\,a_j^{(k)}\,\frac{\partial f}{\partial a_j^{(k)}}, \\
0 = -A_{s+1} f &\equiv \sum_{1}^{\left\{\substack{w-s\\ n}\right.}\!{}_j \sum_{s}^{w-j}\!{}_k \frac{k!\,[k+j+s\,(j-1)]}{(k-s)!\,(s+1)!}\,a_j^{(k-s)}\,\frac{\partial f}{\partial a_j^{(k)}} \\
&\quad - \sum_{1}^{\left\{\substack{w\\ n-1}\right.}\!{}_j \sum_{\left\{\substack{0\\ s-j}\right.}^{w-j}\!{}_k \sum_{\left\{\substack{j-s\\ 0}\right.}^{\left\{\substack{j-1\\ k+j-s}\right.}\!{}_m \frac{j!\,k!\,(n-j)\,a_m^{(k+j-s-m)}}{(j+1-m)!\,m!\,(k+j-s-m)!\,(m+s-j)!}\,\frac{\partial f}{\partial a_j^{(k)}}, \\
&\qquad (s = 1,\ 2, \ldots, w-1) \\
0 = \quad A_{w+1} f &\equiv \sum_{1}^{\left\{\substack{w\\ n-1}\right.}\!{}_j \frac{n-j}{j+1}\,\frac{\partial f}{\partial a_j^{(w-j)}}.
\end{aligned}\right\}\ (29)$$

In using these equations it is to be remembered that $a_0 = 1$, so that $a_0^{(k)} = 0$ when $k > 0$.

In the same manner we get the complete system of

Equations to be satisfied by invariants involving $a_j^{(k)}$ $(k+j \leqq w)$ *and* $y^{(\mu)}$ $(\mu \leqq w)$ *for transformation of both variables* x *and* y.

$$\left.\begin{aligned}
0 &= Y_0 f \equiv \sum_{k=0}^{w} y^{(k)} \frac{\partial f}{\partial y^{(k)}}, \\
0 &= Y_s f \equiv \sum_{k=s}^{w} \frac{k!}{(k-s)!\; s!}\, y^{(k-s)} \frac{\partial f}{\partial y^{(k)}} \\
&\quad - \sum_{j=1}^{\left\{\begin{smallmatrix} w \\ n \end{smallmatrix}\right.} \; \sum_{k=\left\{\begin{smallmatrix} s-j \\ 0 \end{smallmatrix}\right.}^{w-j} \; \sum_{m=\left\{\begin{smallmatrix} 0 \\ j-s \end{smallmatrix}\right.}^{\left\{\begin{smallmatrix} j-1 \\ k+j-1 \end{smallmatrix}\right.} \frac{j!\; k!\; a_m^{(k+j-s-m)}}{(j-m)!\; m!\; (k+j-s-m)!\; (m+s-j)!} \frac{\partial f}{\partial a_j^{(k)}}, \\
&\qquad\qquad (s = 1,\ 2,\ \ldots,\ w) \\
0 &= -X_1 f \equiv \sum_{k=1}^{w} k y^{(k)} \frac{\partial f}{\partial y^{(k)}} + \sum_{j=1}^{\left\{\begin{smallmatrix} w \\ n \end{smallmatrix}\right.} \sum_{k=0}^{w-j} (k+j)\, a_j^{(k)} \frac{\partial f}{\partial a_j^{(k)}}, \\
0 &= -X_{s+1} f \equiv \sum_{k=s+1}^{w} \frac{k!\; y^{(k-s)}}{(k-s-1)!\; (s+1)!} \frac{\partial f}{\partial y^{(k)}} \\
&\quad + \sum_{j=1}^{\left\{\begin{smallmatrix} w-s \\ n \end{smallmatrix}\right.} \sum_{k=s}^{w-j} \frac{k!\; [k+j+s(j-1)]}{(k-s)!\; (s+1)!}\, a_j^{(k-s)} \frac{\partial f}{\partial a_j^{(k)}} \\
&\quad - \sum_{j=1}^{\left\{\begin{smallmatrix} w \\ n-1 \end{smallmatrix}\right.} \; \sum_{k=\left\{\begin{smallmatrix} 0 \\ s-j \end{smallmatrix}\right.}^{w-j} \; \sum_{m=\left\{\begin{smallmatrix} j-s \\ 0 \end{smallmatrix}\right.}^{\left\{\begin{smallmatrix} j-1 \\ k+j-s \end{smallmatrix}\right.} \frac{j!\; k!\; (n-j)\, a_m^{(k+j-s-m)}}{(j+1-m)!\; m!\; (k+j-s-m)!\; (m+s-j)!} \frac{\partial f}{\partial a_j^{(k)}}, \\
&\qquad\qquad (s = 1,\ 2,\ \ldots,\ w-1) \\
0 &= X_{w+1} f \equiv \sum_{j=1}^{\left\{\begin{smallmatrix} w \\ n-1 \end{smallmatrix}\right.} \frac{n-j}{j+1} \frac{\partial f}{\partial a_j^{(w-j)}}.
\end{aligned}\right\} \quad (30)$$

§27. Using the notation of (30) we may write (28) in the form

$$X^{(ww)} f = \xi \frac{\partial f}{\partial x} + \sum_{s=0}^{w} \phi^{(s)} Y_s f + \sum_{s=1}^{w+1} \xi^{(s)} X_s f. \qquad (31)$$

The equations $Y_s f = 0$ in (30) are, of course, the same as the equations (9) which yield the covariants for transformation of y alone. The equations $X_s f = 0$ in (30) yield the covariants for transformation of x alone.

§28. The equation $0 = -A_1 f$ of the system (29) shows that every solution of this system must be an isobaric function of the $a_j^{(k)}$'s of weight zero. Every rational invariant must therefore be the quotient of two integral rational functions, which are isobaric of the same weight. The equation $0 = -X_1 f$ of (30) shows that the covariants must also be isobaric of weight zero in the $a_j^{(k)}$'s and $y^{(\mu)}$'s, and the equation $Y_0 f = 0$ that they must be homogeneous of the zeroth degree in $y, y', \ldots\ldots, y^{(w)}$. (Cf. Theorems I and III.)

§29. The complete system of partial differential equations (29) contains $2w + 1$ equations. When $w \leqq n$ there are $\frac{1}{2} w (w + 1)$ variables. Hence there are at least

$$\tfrac{1}{2} w (w + 1) - (2w + 1) = \tfrac{1}{2} (w^2 - 3w - 2)$$

solutions. In order to have any solutions for this case we must have $n \geqq w \geqq 4$. When $w > n$ there are $nw - \frac{1}{2} n (n - 1)$ variables, and therefore

$$nw - \tfrac{1}{2} n (n - 1) - (2w + 1) = (n - 2) w - \tfrac{1}{2} (n^2 - n - 2)$$

solutions. When $n = 2$ this is always negative, however great w be chosen, which shows, if we assume the equations to be independent, that the linear differential equation of the second order has no invariants, a fact which has long been known. When $n = 3$ there are $w - 4$ solutions, so that there is certainly a solution for $w = 5$, that is, involving terms of weight five. We shall return to this later. Similarly we see that there is one solution involving only a_1, a_2, a_3 and their derivatives to those of weight 5, whatever n may be. Call this invariant I_3. Either its numerator or denominator involve terms of weight 5. Then making $w = 4$ in the formula $\frac{1}{2} (w^2 - 3w - 2)$ for the number of solutions, we see that there is one solution which must involve a_4. Call this solution I_4. When $w = 5$ there are four solutions, one of which involves a_5, and which we shall call I_5. And in general call that solution I_m which involves a_m and a's of smaller subscript, with their derivatives up to those of weight m. When $w \leqq m$ we can in this manner pick out a set of $w - 2$ independent solutions, from which we may obtain the rest by differentiation. These solutions which we have just chosen will be algebraic, and in all of the examples which follow they may be made rational by simply raising to a power. If we assume then that

$$I_m = \frac{U^{(m)}}{V^{(m)}},$$

(21) shows that

$$\Delta_m I_m = \left[\frac{1}{V^{(m)}} \right]^{\frac{1}{m}} \frac{d}{dx} I_m$$

is an invariant, and

$$I_m, \; \Delta_m I_m, \; \Delta_m^2 I_m, \ldots\ldots, \Delta_m^{w-m} I_m; \qquad (m = 4, 5, \ldots, w)$$

is a set of $w - m + 1$ invariants, all independent, and none involving terms of weight greater than w. The invariant I_3 is not included in this set, and can be differentiated only $w - 5$ times, since it involves terms of weight 5, not 3. It therefore furnishes only $w - 4$ solutions. By subjecting

$$I_3 ; I_4, I_5, \ldots, I_w$$

to this process we thus get

$$w - 4 + [(w - 3) + (w - 4) + (w - 5) + \ldots + 2 + 1] = \tfrac{1}{2}(w^2 - 3w - 2)$$

invariants which are evidently independent; that is, this process gives all the invariants. The system of $w - 2$ invariants

$$I_3, I_4, \ldots, I_w$$

is then a *complete system of invariants* from which all others may be derived by differentiation.

When $w > n$ the same process applied to

$$I_3, I_4, \ldots, I_n$$

yields

$$w - 4 + [(w - 3) + (w - 4) + \ldots + (w - n + 1)] = (n - 2)w - \tfrac{1}{2}(n^2 - n + 2)$$

invariants, and we have here all the solutions.

§30. The complete system of partial differential equations (30) contains $2w + 2$ equations. When $w < n$ the number of variables is $w + 1 + \frac{1}{2}w(w + 1)$, and hence there are at least $\frac{1}{2}(w - 2)(w + 1)$ independent solutions. Of these solutions $\frac{1}{2}(w^2 - 3w - 2)$ are solutions of (29), so that we have still w to find. An examination of (30) shows that there is one solution involving y and y' and a's of weight not greater than 4. The differentiation process may be applied $w - 4$ times to this covariant, which gives us all but the three covariants of highest order. The process cannot be applied oftener because it would then bring in a's of weight greater than w. We shall presently give the solution of (30) for $w = 3$. This gives two covariants involving y, y', y'', y''' and a's of weight not greater than 3. The application of the differentiation process to these two yields the whole series of covariants except the one containing only y and y', which must be determined by a separate integration. This latter covariant is also given later.

§31. When, in (30), $w \leqq n$, there are $nw - \frac{1}{2}n(n - 1) + w + 1$ variables, but they are not all independent, being bound by the given differential equation (2), so that $y^{(n)}$ may always be eliminated from a covariant of order in y equal to or greater than n. Moreover, the covariants of order in y higher than n have little interest, for they can always be found by a differentiation process. The given equation (2) is an invariant equation, and, as will be shown later.

$$2a_3 - 3a_2' - 6a_1a_2 + 4a_1^3 + 6a_1a_1' + a_1''$$

is also a relative invariant. Hence by §17

$$y^{(n)} + na_1y^{(n-1)} + \frac{n(n-1)}{2}a_2y^{(n-2)} + \ldots + a_ny$$

is an absolute covariant (equal to zero when we use (2)). The repeated application of (2) to this function will give *all* the covariants of order in y greater than n. The quantity $y^{(n)}$ must then be eliminated from these by means of (2). We shall leave these covariants out of consideration, and when $w_2 \leqq n$, take $w_1 = n-1$. These values in (28) show that these covariants are the solutions of the following complete system:

$$\left.\begin{aligned}
0 = \quad Y_0 f \quad &\equiv \sum_{0}^{n-1}{}_k\, y^{(k)} \frac{\partial f}{\partial y^{(k)}}, \\
0 = \quad Y_s f \quad &\equiv \sum_{s}^{n-1}{}_k \frac{k!}{(k-s)!\, s!} y^{(k-s)} \frac{\partial f}{\partial y^{(k)}} \\
&\quad - \sum_{1}^{n}{}_j \sum_{\left\{\begin{smallmatrix} s-j \\ 0 \end{smallmatrix}\right.}^{w-j}{}_k \sum_{\left\{\begin{smallmatrix} 0 \\ j-s \end{smallmatrix}\right.}^{\left\{\begin{smallmatrix} j-1 \\ k+j-s \end{smallmatrix}\right.}{}_m \frac{j!\, k!\, a_m^{(k+j-s-m)}}{(j-m)!\, m!\, (k+j-s-m)!\, (m+s-j)!} \frac{\partial f}{\partial a_j^{(k)}}, \\
&\qquad (s = 1, 2, \ldots, n-1) \\
0 = -Y_s f \quad &\equiv \sum_{1}^{n}{}_j \sum_{s-j}^{w-j}{}_k \sum_{0}^{\left\{\begin{smallmatrix} j-1 \\ k+j-s \end{smallmatrix}\right.}{}_m \frac{j!\, k!\, a_m^{(k+j-s-m)}}{(j-m)!\, m!\, (k+j-s-m)!\, (m+s-j)!} \frac{\partial f}{\partial a_j^{(k)}}, \\
&\qquad (s = n, n+1, \ldots, w) \\
0 = -X_1 f \quad &\equiv \sum_{1}^{n-1}{}_k\, k y^{(k)} \frac{\partial f}{\partial y^{(k)}} + \sum_{1}^{n}{}_j \sum_{0}^{w-j}{}_k (k+j)\, a_j^{(k)} \frac{\partial f}{\partial a_j^{(k)}}, \\
0 = -X_{s+1} f &\equiv \sum_{s+1}^{n-1}{}_k \frac{k!\, y^{(k-s)}}{(k-s-1)!\, (s+1)!} \frac{\partial f}{\partial y^{(k)}} \\
&\quad + \sum_{1}^{\left\{\begin{smallmatrix} n \\ w-s \end{smallmatrix}\right.}{}_j \sum_{s}^{w-j}{}_k \frac{k!\, [k+j+s(j-1)]}{(k-s)!\, (s+1)!} a_j^{(k-s)} \frac{\partial f}{d a_j^{(k)}} \\
&\quad - \sum_{1}^{n-1}{}_j \sum_{\left\{\begin{smallmatrix} 0 \\ s-j \end{smallmatrix}\right.}^{w-j}{}_k \sum_{\left\{\begin{smallmatrix} j-s \\ 0 \end{smallmatrix}\right.}^{\left\{\begin{smallmatrix} j-1 \\ k+j-s \end{smallmatrix}\right.}{}_m \frac{j!\, k!\, (n-j)\, a_m^{(k+j-s-m)}}{(j+1-m)!\, m!\, (k+j-s-m)!\, (m+s-j)!} \frac{\partial f}{\partial a_j^{(k)}}, \\
&\qquad (s = 1, 2, \ldots, n-2) \\
0 = -X_{s+1} f &\equiv \sum_{1}^{\left\{\begin{smallmatrix} n \\ w-s \end{smallmatrix}\right.}{}_j \sum_{s}^{w-j}{}_k \frac{k!\, [k+j+s(j-1)]}{(k-s)!\, (s+1)!} a_j^{(k-s)} \frac{\partial f}{\partial a_j^{(k)}} \\
&\quad - \sum_{1}^{n-1}{}_j \sum_{s-j}^{w-j}{}_k \sum_{0}^{\left\{\begin{smallmatrix} j-1 \\ k+j-s \end{smallmatrix}\right.}{}_m \frac{j!\, k!\, (n-j)\, a_m^{(k+j-s-m)}}{(j+1-m)!\, m!\, (k+j-s-m)!\, (m+s-j)!} \frac{\partial f}{\partial a_j^{(k)}}, \\
&\qquad (s = n-1, n, \ldots, w-1) \\
0 = X_{w+1} f &\equiv \sum_{1}^{n-1}{}_j \frac{n-j}{j+1} \frac{\partial f}{\partial a_j^{(w-j)}}.
\end{aligned}\right\} (32)$$

The number of solutions of these equations is

$$nw-\tfrac{1}{2}n(n-3)-(2w+2)=(n-2)w-\tfrac{1}{2}(n^2-3n+4).$$

Of these all but $n-1$ are solutions of (29). These $n-1$ are found just as for the case $w<n$, and in fact are the same solutions as for $w=n-1$.

§32. As an example, let us give the solutions of (30) for $w=3$. The equations become:

$$\begin{aligned}
0&= \quad Y_0f\equiv y\frac{\partial f}{\partial y}+y'\frac{\partial f}{\partial y'}+y''\frac{\partial f}{\partial y''}+y'''\frac{\partial f}{\partial y'''},\\
0&= \quad Y_1f\equiv y\frac{\partial f}{\partial y'}+2y'\frac{\partial f}{\partial y''}+3y''\frac{\partial f}{\partial y'''}-\frac{\partial f}{\partial a_1}\\
&\qquad\qquad -2a_1\frac{\partial f}{\partial a_2}-2a_1'\frac{\partial f}{\partial a_2'}-3a_2\frac{\partial f}{\partial a_3},\\
0&= \quad Y_2f\equiv y\frac{\partial f}{\partial y''}+3y'\frac{\partial f}{\partial y'''}-\frac{\partial f}{\partial a_1'}-\frac{\partial f}{\partial a_2}-2a_1\frac{\partial f}{\partial a_2'}-3a_1\frac{\partial f}{\partial a_3},\\
0&= \quad Y_3f\equiv y\frac{\partial f}{\partial y'''}-\frac{\partial f}{\partial a_1''}-\frac{\partial f}{\partial a_2'}-\frac{\partial f}{\partial a_3},\\
0&=-X_1f\equiv y'\frac{\partial f}{\partial y'}+2y''\frac{\partial f}{\partial y''}+3y'''\frac{\partial f}{\partial y'''}+a_1\frac{\partial f}{\partial a_1}+2a_1'\frac{\partial f}{\partial a_1'}\\
&\qquad\qquad +3a_1''\frac{\partial f}{\partial a_1''}+2a_2\frac{\partial f}{\partial a_2}+3a_2'\frac{\partial f}{\partial a_2'}+3a_3\frac{\partial f}{\partial a_3},\\
0&=-X_2f\equiv y'\frac{\partial f}{\partial y''}+3y''\frac{\partial f}{\partial y'''}-\frac{n-1}{2}\frac{\partial f}{\partial a_1}+a_1\frac{\partial f}{\partial a_1'}+3a_1'\frac{\partial f}{\partial a_1''}\\
&\qquad\qquad -(n-2)a_1\frac{\partial f}{\partial a_2}+[2a_2-(n-2)a_1']\frac{\partial f}{\partial a_2'}-\frac{3(n-3)}{2}a_2\frac{\partial f}{\partial a_3},\\
0&=-X_3f\equiv y'\frac{\partial f}{\partial y'''}-\frac{n-1}{2}\frac{\partial f}{\partial a_1'}+a_1\frac{\partial f}{\partial a_1''}-\frac{n-2}{3}\frac{\partial f}{\partial a_2}\\
&\qquad\qquad -(n-2)a_1\frac{\partial f}{\partial a_2'}-(n-3)a_1\frac{\partial f}{\partial a_3},\\
0&= \quad X_4f\equiv\frac{n-1}{2}\frac{\partial f}{\partial a_1''}+\frac{n-2}{3}\frac{\partial f}{\partial a_2'}+\frac{n-3}{4}\frac{\partial f}{\partial a_3}.
\end{aligned}$$

This complete system of 8 equations (which are independent) in 10 variables has two solutions. By the ordinary methods they are found to be

$$\frac{(n-1)yy''-(n-2)y'^2+2a_1yy'+\dfrac{(n-1)^2}{n+1}\left[3a_2-\dfrac{2(n-2)}{n-1}a_1'-\dfrac{(n-2)(3n-1)}{(n-1)^2}a_1^2\right]y^2,}{y^2(2a_3-3a_2'-6a_2a_1+4a_1^3+6a_1a_1'+a_1'')^{\frac{2}{3}}}$$

and

$$\frac{\left[y''' + \frac{6}{n-1}a_1y'' + \frac{1\cdot 2}{n+1}a_2y' + \frac{2(n-1)}{n+1}a_3y\right]y^2 + (n-3)\left\{\begin{array}{l} \frac{-3}{n-1}y''y'y + \frac{2(n-2)}{(n-1)^2}y'^2 - \frac{6}{(n-1)^2}a_1y'^2y \\ \qquad\qquad + \frac{6}{n^2-1}a_1'y'y^2 + \frac{12n}{(n-1)^2(n+1)}a_1^2y'y^2 \\ \quad + \left[4\frac{n^2-3n+1}{(n-1)^2(n+1)}a_1^3 - \frac{6}{n+1}a_2a_1 - \frac{6}{n^2-1}a_1a_1' - \frac{1}{n+1}a_1''\right]y^3 \end{array}\right\}}{y^3(2a_3 - 3a_2' - 6a_2a_1 + 4a_1^3 + 6a_1a_1' + a_1'')}.$$

Both numerator and denominator of both of these invariants are relative invariants (Th. III), and this justifies the use of the denominator made in §31. When $n = 3$, the numerator of the second invariant reduces to

$$(y''' + 3a_1y'' + 3a_2y' + a_3y)\,y^2,$$

as it should.

These are the two covariants mentioned in §30 from which all others (except the one involving only y and y') may be derived by differentiation.

The covariant involving only y and y' is the solution of the following complete system, which is obtained by writing (30) for $w = 4$ and then dropping the derivatives with regard to y'', y''', y^{IV} and a_4.

$$0 = y\frac{\partial f}{\partial y} + y'\frac{\partial f}{\partial y'},$$

$$0 = y\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial a_1} - 2a_1\frac{\partial f}{\partial a_2} - 2a_1'\frac{\partial f}{\partial a_2'} - 2a_1''\frac{\partial f}{\partial a_2''} - 3a_2\frac{\partial f}{\partial a_3} - 3a_2'\frac{\partial f}{\partial a_3'},$$

$$0 = \frac{\partial f}{\partial a_1'} + \frac{\partial f}{\partial a_2} + 2a_1\frac{\partial f}{\partial a_2'} + 4a_1'\frac{\partial f}{\partial a_2''} + 3a_1\frac{\partial f}{\partial a_3} + 3(a_2 + a_1')\frac{\partial f}{\partial a_3'},$$

$$0 = \frac{\partial f}{\partial a_1''} + \frac{\partial f}{\partial a_2'} + \frac{\partial f}{\partial a_3} + 3a_1\frac{\partial f}{\partial a_3'},$$

$$0 = \frac{\partial f}{\partial a_1'''} + \frac{\partial f}{\partial a_2''} + \frac{\partial f}{\partial a_3'},$$

$$0 = y'\frac{\partial f}{\partial y'} + a_1\frac{\partial f}{\partial a_1} + 2a_1'\frac{\partial f}{\partial a_1'} + 3a_1''\frac{\partial f}{\partial a_1''} + 4a_1'''\frac{\partial f}{\partial a_1'''} + 2a_2\frac{\partial f}{\partial a_2} + 3a_2'\frac{\partial f}{\partial a_2'}$$
$$+ 4a_2''\frac{\partial f}{\partial a_2''} + 3a_3\frac{\partial f}{\partial a_3} + 4a_3'\frac{\partial f}{\partial a_3'},$$

$$0 = -\frac{n-1}{2}\frac{\partial f}{\partial a_1} + a_1\frac{\partial f}{\partial a_1'} + 3a_1'\frac{\partial f}{\partial a_1''} + 6a_1''\frac{\partial f}{\partial a_1'''} - (n-2)a_1\frac{\partial f}{\partial a_2}$$
$$+ [2a_2 - (n-2)a_1']\frac{\partial f}{\partial a_2'} + [5a_2' - (n-2)a_1'']\frac{\partial f}{\partial a_2''}$$
$$- \frac{3(n-3)}{2}a_2\frac{\partial f}{\partial a_3} + \left[3a_3 - \frac{3(n-3)}{2}a_2'\right]\frac{\partial f}{\partial a_3'},$$

$$0 = -\frac{n-1}{2}\frac{\partial f}{\partial a_1'} + a_1\frac{\partial f}{\partial a_1''} + 4a_1'\frac{\partial f}{\partial a_1'''} - \frac{n-2}{3}\frac{\partial f}{\partial a_2} - (n-2)a_1\frac{\partial f}{\partial a_2'}$$
$$+ 2[a_2 - (n-2)a_1']\frac{\partial f}{\partial a_2''} - (n-3)a_1\frac{\partial f}{\partial a_3} - (n-3)(\tfrac{3}{2}a_2 + a_1')\frac{\partial f}{\partial a_3'},$$

$$0 = -\frac{n-1}{2}\frac{\partial f}{\partial a_1''} + a_1\frac{\partial f}{\partial a_1'''} - \frac{n-2}{3}\frac{\partial f}{\partial a_2'} - (n-2)a_1\frac{\partial f}{\partial a_2''}$$
$$- \frac{n-3}{4}\frac{\partial f}{\partial a_3} - (n-3)a_1\frac{\partial f}{\partial a_3'},$$

$$0 = \frac{n-1}{2}\frac{\partial f}{\partial a_1'''} + \frac{n-2}{3}\frac{\partial f}{\partial a_2''} + \frac{n-3}{4}\frac{\partial f}{\partial a_3'}.$$

The one solution of this system of equations is

$$\frac{\begin{array}{l}6(y' + a_1y)(2a_3 - 3a_2' - 6a_2a_1 + 4a_1^3 + 6a_1a_1' + a_1'') \\ \qquad + (n-1)y(2a_3' - 3a_2'' - 6a_2'a_1 - 6a_2a_1' + 12a_1^2a_1' + 6a_1a_1'' + 6a_1'^2 + a_1''')\end{array}}{y(2a_3 - 3a_2' - 6a_2a_1 + 4a_1^3 + 6a_1a_1' + a_1'')^{\frac{4}{3}}}.$$

CHAPTER 4.

Consideration of a Subgroup.

§33. Let us now turn to the consideration of a case which within the last few years has received the attention of a number of writers, and which was perhaps first completely treated by Forsyth.* If in the equation

$$y^{(n)} + na_1y^{(n-1)} + \dots + a_ny = 0 \tag{2}$$

we make the transformation

$$y = ze^{-\int a_1\,dx},$$

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* Philosophical Transactions, 1888, I, "On Invariants, etc., associated with the Linear Differential Equation," pp. 377–489.

the term involving $z^{(n-1)}$ disappears, and we get

$$z^{(n)}+\frac{n(n-1)}{2}p_2 z^{(n-2)}+\ldots\ldots+p_n z=0, \tag{33}$$

where $p_2, p_3, \ldots\ldots p_n$ are functions of x alone. If in this equation x be subjected to an arbitrary transformation

$$x_1=\chi(x),$$

we can afterwards cause the term involving the $(n-1)$th derivative to disappear by a transformation of the form

$$z_1=z\Psi(x).$$

All such transformations must form a subgroup of the group (11), for they are included in it and leave the form of (33) unaltered. We may then seek the invariants of (33) for the group of transformations of this form. The infinitesimal transformations of this subgroup are readily found. From (27) we have

$$\delta a_1=-\delta t\left[\xi' a_1+\phi'-\frac{n-1}{2}\xi''\right].$$

In the case under consideration $a_1=0$ and $\delta a_1=0$. Hence we must have the relation

$$\phi'=\frac{n-1}{2}\xi''. \tag{34}$$

This may be taken as the defining equation of the subgroup in question. Integrated, it is

$$\phi=\frac{n-1}{2}\xi'+C, \qquad (C=\text{constant})$$

§34. When the preceding value of ϕ is substituted in (24), the result is

$$\delta z^{(k)}=\left[Cz^{(k)}+\tfrac{1}{2}\sum_0^k{}_m\frac{k!\;[(m+1)(n+1)-2(k+1)]}{(k-m)!\;(m+1)!}\xi^{(m+1)}z^{(k-m)}\right]\delta t. \tag{35}$$

Similarly (26) becomes

$$\frac{\delta p_j^{(k)}}{\delta t}=-\sum_0^k{}_s\frac{k!\;[j(k+1)-s(j-1)]}{(k-s+1)!\;s!}p_j^{(s)}\xi^{(k-s+1)}$$
$$-\tfrac{1}{2}\sum_0^{j-1}{}_s\frac{j!\;[n(j-s-1)+j+s-1]}{(j-s+1)!\;s!}\frac{d^k}{dx^k}(\xi^{(j-s+1)}p_s). \tag{36}$$

$$(j=2,3,\ldots\ldots,n;\; k=0,1,2,\ldots\ldots;\; p_0=1,\; p_1=0)$$

This is Forsyth's (13). Our (27) becomes

$$\frac{\delta p_j^{(k)}}{\delta t} = -\sum_{s=1}^{k+1} \frac{k!\,[k+1+s(j-1)]}{(k-s+1)!\,s!} p_j^{(k-s+1)}\xi^{(s)}$$

$$-\tfrac{1}{2}\sum_{s=2}^{k+j+1}\xi^{(s)}\sum_{m=\left\{\begin{smallmatrix}j+1-s\\0\end{smallmatrix}\right.}^{\left\{\begin{smallmatrix}j-1\\k+j+1-s\end{smallmatrix}\right.} \frac{j!\,k!\,[n(j-m-1)+j+m-1]\,p_m^{(k+j+1-s-m)}}{(j+1-m)!\,m!\,(k+j+1-s-m)!\,(m+s-j-1)!}. \qquad (37)$$

$$(j=2,\,3,\,\ldots,\,n;\ k=0,\,1,\,2,\,\ldots\ ;\ p_0=1,\ p_1=0)$$

For $j=1$, (37) gives

$$\frac{\delta p_1^{(k)}}{\delta t} = -\sum_{s=1}^{k+1}\frac{(k+1)!}{(k-s+1)!\,s!}p_1^{(k-s+1)}\xi^{(s)} \equiv 0,\ \text{since}\ p_1 \equiv 0.$$

§35. The extended transformation, corresponding to (28), is

$$Z^{(w_1 w_2)}f = \xi\frac{\partial f}{\partial x} + C\sum_{k=0}^{w_1} z^{(k)}\frac{\partial f}{\partial z^{(k)}} + \tfrac{1}{2}\sum_{s=1}^{w_1+1}\xi^{(s)}\sum_{k=s-1}^{w_1}\frac{k!\,[s(n+1)-2(k+1)]}{s!\,(k-s+1)!}z^{(k-s+1)}\frac{\partial f}{\partial z^{(k)}}$$

$$-\sum_{s=1}^{w_2-1}\xi^{(s)}\sum_{j=2}^{\left\{\begin{smallmatrix}n\\w_2-s+1\end{smallmatrix}\right.}\sum_{k=s-1}^{w_2-j}\frac{k!\,[k+1+s(j-1)]}{(k-s+1)!\,s!}p_j^{(k-s+1)}\frac{\partial f}{\partial p_j^{(k)}} \qquad (38)$$

$$-\tfrac{1}{2}\sum_{s=2}^{w_2+1}\xi^{(s)}\sum_{j=2}^{\left\{\begin{smallmatrix}n\\w_2\end{smallmatrix}\right.}\sum_{k=\left\{\begin{smallmatrix}s-j-1\\0\end{smallmatrix}\right.}^{w_2-j}\sum_{m=\left\{\begin{smallmatrix}0\\j+1-s\end{smallmatrix}\right.}^{\left\{\begin{smallmatrix}j-1\\k+j+1-s\end{smallmatrix}\right.}\frac{j!\,k!\,[n(j-m-1)+j+m-1]\,p_m^{(k+j+1-s-m)}}{(j+1-m)!\,m!\,(k+j+1-s-m)!\,(m+s-j-1)!}\frac{\partial f}{\partial p_j^{(k)}},$$

To get the equations whose solutions are the invariants we must equate the coefficients of $\xi^{(s)}$, and also that of C, to zero. As before, the two cases $w_1=0$, $w_2=w$ and $w_1=w_2=w$ are considered. We find thus the following complete systems of equations:

Equations to be satisfied by invariants involving only $p_j^{(k)}$ $(k+j \leq w)$, *for the sub-group,* $\phi' = \frac{n-1}{2}\xi''$.

$$\left.\begin{aligned}
0 &= -A_1'f \equiv \sum_{j=2}^{\left\{\substack{w\\n}\right.} \sum_{k=0}^{w-j} (k+j)\, p_j^{(k)} \frac{\partial f}{\partial p_j^{(k)}},\\
0 &= -2A_{s+1}'f \equiv 2\sum_{j=2}^{\left\{\substack{n\\w-s}\right.} \sum_{k=s}^{w-j} \frac{k!\,[k+j+s(j-1)]}{(k-s)!\,(s+1)!}\, p_j^{(k-s)} \frac{\partial f}{\partial p_j^{(k)}}\\
&\quad + \sum_{j=2}^{\left\{\substack{n\\w}\right.} \sum_{k=\left\{\substack{0\\s-j}\right.}^{w-j} \sum_{m=\left\{\substack{j-s\\0}\right.}^{\left\{\substack{j-1\\k+j-s}\right.} \frac{j!\,k!\,[n(j-m-1)+j+m-1]\,p_m^{(k+j-s-m)}}{(j+1-m)!\,m!\,(k+j-s-m)!\,(m+s-j)!} \frac{\partial f}{\partial p_j^{(k)}},\\
&\qquad (s = 1, 2, \ldots, w-2)\\
0 &= -\frac{2}{n+1} A_{w+1-s}'f \equiv \sum_{j=2}^{\left\{\substack{n\\w-s}\right.} \frac{j-1}{j+1} \frac{\partial f}{\partial p_j^{(w-j-s)}}. \qquad (s = 0, 1)
\end{aligned}\right\} (39)$$

Equations to be satisfied by invariants involving $p_j^{(k)}$, $(k+j \leq w)$ *and* $z^{(l)}$ $(l \leq w)$, *for the subgroup* $\phi' = \frac{n-1}{2}\xi''$.

$$\left.\begin{aligned}
0 &= \quad Y_0'f \equiv \sum_{k=0}^{w} z^{(k)} \frac{\partial f}{\partial z^{(k)}},\\
0 &= -\ X_1'f + \frac{n-1}{2} Y_0'f \equiv \sum_{k=1}^{w} k z^{(k)} \frac{\partial f}{\partial z^{(k)}} - A_1'f,\\
0 &= \quad 2X_{s+1}'f \equiv \sum_{k=s}^{w} \frac{k!\,[s(n+1)+n-2k-1]}{(s+1)!\,(k-s)!}\, z^{(k-s)} \frac{\partial f}{\partial z^{(k)}} + 2A_{s+1}'f,\\
&\qquad (s = 1, 2, \ldots, w)
\end{aligned}\right\} (40)$$

In making use of (40) the value of $A_{s+1}'f$ must be substituted from (39), and in using either (39) or (40) we must remember that $p_0 = 1$, $p_1 = 0$.

§36. The complete system of partial differential equations (39) contains $w+1$ equations, and, when $w \leq n$, $\frac{1}{2}w(w-1)$ variables. It has therefore at least $\frac{1}{2}(w^2 - 3w - 2)$ solutions. These are of course the $\frac{1}{2}(w^2 - 3w - 2)$ solutions of (29) for $a_1^{(k)} = 0$, $a_j^{(k)} = p_j^{(k)}$ $(j > 1)$. Similarly when $w > n$ there are $(n-2)w - \frac{1}{2}(n^2 - n + 2)$ solutions, which are obtained from the $(n-2)w$

$-\frac{1}{2}(n^2 - n + 2)$ solutions of (29) for $w > n$ by writing $a_1^{(k)} = 0$, $a_j^{(k)} = p_j^{(k)}$. Or the solutions may be chosen as indicated in §29.

The system (40) has $w + 2$ equations, and, when $w < n$, $\frac{1}{2}w(w-1) + w + 1$ variables, and therefore $\frac{1}{2}(w+1)(w-2)$ solutions. When $w \leqq n$, the treatment is exactly the same as for the general case (§30), but it is not thought to be necessary to write out the equations corresponding to (32). They are obtained from (40) by neglecting those terms in which $\frac{\partial f}{\partial z^{(k)}}$ occurs, where $k \leqq n$. The first two equations of (40) show that every absolute invariant must be homogeneous of degree zero in $z, z', \ldots\ldots, z^{(w)}$, and isobaric of weight zero, as in the general case.

The Relative Invariants.

§37. Relative invariants, that is, invariant equations, may be formed either by equating an absolute invariant to a constant, or by forming determinants of the matrix of the coefficients of (39) or (40), as indicated by the general theory. This latter process is long, and the writer has found no method of applying it to the general case.

§38. We may use another method to find the relative invariants which are reproduced save as to a factor of known form, and §16 shows us that the form of the factor is *always* known. Let us seek, as Forsyth does, those relative invariants, V, for which

$$V_1 = \left(\frac{d\chi}{dx}\right)^{\nu} V, \tag{41}$$

where ν is an integer. Such invariants V cannot be the *absolute* invariants of any but a trivial subgroup of (34). For suppose that V were an absolute invariant. Then

$$\frac{d\chi}{dx} = 1^{\frac{1}{\nu}}$$

and

$$x_1 = \chi(x) = 1^{\frac{1}{\nu}} x + \text{const.}$$

As we are dealing with groups containing the identical transformation, this becomes

$$x_1 = x + \text{const.},$$

$$\delta x_1 = \delta t,$$

$$\xi = 1, \quad \xi' = 0, \quad \phi = C, \quad \delta y = Cy\,\delta t \text{ (by (34))}.$$

(37) then gives $\delta p_j^{(k)} = 0$. $p_j^{(k)}$ is therefore not transformed, and *every* function of $p_j^{(k)}$ is an invariant for this transformation

$$\begin{cases} x_1 = x + \alpha, \\ y_1 = \beta y. \end{cases}$$

The relative invariants in question cannot, therefore, be found as the absolute invariants of *any subgroup* of (34).

§39. They may be found, however, as the solutions of certain partial differential equations. If in (41) we write

$$\chi = x + \delta t . \xi(x),$$

so as to get the infinitesimal transformation, we find

$$\begin{aligned} V_1 &= (1 + \xi' \delta t)^{\nu} V, \\ \delta V &= \nu \xi' V . \delta t = Z^{(w_1 w_2)} V . \delta t \text{ by } (38), \\ \therefore\ Z^{(w_1 w_2)} V &= \nu V . \xi'. \end{aligned}$$

This equation must subsist identically for every ξ. It therefore breaks up into a number of equations, which are the same as (40) with the exception of the equation obtained by equating the coefficient of ξ' to zero. This equation would be replaced by

$$\nu f = \sum_{1}^{w} {}^{k}\, k z^{(k)} \frac{\partial f}{\partial z^{(k)}} + \sum_{2}^{\left\{ w \atop n \right.} {}^{j} \sum_{0}^{w-j} {}^{k} (k + j) p_j^{(k)} \frac{\partial f}{\partial p_j^{(k)}} \equiv W f.$$

This equation tells us that f must be an isobaric function of weight ν (cf. Theorem I). If f be isobaric of weight ν, then $f^{\frac{\mu}{\nu}}$ is isobaric of weight μ, and is a solution of the other equations if f is. The ν of the foregoing equation may therefore be replaced by any other number, zero excepted, for instance by $\nu = 1$. The solutions seem to be simplest when we set $\nu = w$. We have then

Equations to be satisfied by the relative invariants for which $V_1 = \chi'^{w} V$, *for the subgroup*, $\phi' = \frac{n-1}{2} \xi''$.

$$\left.\begin{aligned} w f &= \sum_{1}^{w} {}^{k}\, k z^{(k)} \frac{\partial f}{\partial z^{(k)}} + \sum_{2}^{\left\{ w \atop n \right.} {}^{j} \sum_{0}^{w-j} {}^{k} (k + j) p_j^{(k)} \frac{\partial f}{\partial p_j^{(k)}}, \\ 0 &= \sum_{0}^{w} {}^{k}\, z^{(k)} \frac{\partial f}{\partial z^{(k)}}, \\ 0 &= 2 X'_{s+1} f, \qquad (s = 1, 2, \ldots, w) \end{aligned}\right\} \qquad (42)$$

where $X'_{s+1} f$ is given by (40).

§40. If these equations be rendered homogeneous by substituting a relation of the form

$$F(f, \ldots . z^{(l)}, \ldots . p_j^{(k)} \ldots .) = 0,$$

it is readily shown that the resulting system of equations form a complete system with $w + 2$ equations and one more variable than (40). It has therefore one more solution than (40). All but one of these solutions may be chosen free from f, being the solutions of (40). One solution F must contain f, and therefore there is at least one solution of the system (42). If this solution be multiplied by any solution of (40) we evidently get a new solution of (42). Conversely, the quotient of any two solutions of (42) is a solution of (40). These two facts together show that the number of solutions of (42) is exactly one greater than the number of solutions of (40), and *all* the solutions of (40) may be expressed as quotients of solutions of (42). (Cf. Theorem III.)

§41. The number of solutions of (42) which are free from $z^{(l)}$ is one greater than the number of solutions of (39), therefore at least $\frac{1}{2} w (w - 3)$. When $w = 3$ the equations obtained from (42) are not independent, there is one relation between them. They are

$$\begin{aligned} 3f &= \quad 2p_2 \frac{\partial f}{\partial p_2} + 3p_2' \frac{\partial f}{\partial p_2'} + 3p_3 \frac{\partial f}{\partial p_3}, \\ 0 &= -4p_2 \frac{\partial f}{\partial p_2'} - 6p_2 \frac{\partial f}{\partial p_3}, \\ 0 &= -\frac{n+1}{3} \frac{\partial f}{\partial p_2}, \\ 0 &= -\frac{n+1}{3} \frac{\partial f}{\partial p_2'} - \frac{n+1}{2} \frac{\partial f}{\partial p_3}. \end{aligned}$$

The one solution is $f = p_3 - \frac{3}{2} p_2'$. The other solutions of (42) for $w = 3$ are

$$\frac{1}{z^3} \left[(n-1) z''z - (n-2) z'^2 + \frac{3(n-1)^2}{n+1} p_2 z^2 \right]^{\frac{3}{2}},$$

$$\frac{1}{z^3} \Big[(n-1)^2 z'''z - 3(n-1)(n-3) z'' z' z + 2(n-2)(n-3) z'^3$$
$$+ \frac{12(n-1)^2}{n+1} p_2 z' z^2 + \frac{2(n-1)^3}{n+1} p_3 z^3 \Big].$$

Either of the last two solutions divided by $p_3 - \frac{3}{2} p_1'$ is a solution of (40), and of course may be obtained from the solutions of (30) found in §32 by setting $a_1 = 0$.

CHAPTER 5.

Differential Equation in Canonical Form.

§42. Laguerre has shown* that the general equation (1) may, by suitably choosing the functions χ and Ψ of the transformation (11), always be transformed into an equation in which $a_1 = a_2 = 0$. Let us take this, with Forsyth, as the canonical form of (1). We then have to find the invariants of

$$u^{(n)} + \sum_{3}^{n} \frac{n!}{(n-s)!\,s!}\, q_s u^{(n-s)} = 0. \qquad (43)$$

The transformations which leave (43) invariant must be among those defined by (34), since $q_1 = 0$. Therefore we may use (37) to compute δp_2. It gives

$$\delta p_2 = -\delta t\left[2p_2\xi' + \frac{n+1}{6}\,\xi'''\right].$$

This must vanish when $p_2 = 0$. Hence the transformations which leave (43) invariant are defined by

$$\xi''' = 0, \quad \phi' = \frac{n-1}{2}\,\xi'', \qquad (44)$$

or $\qquad \delta x = (\alpha_0 + 2\alpha_1 x + \alpha_2 x^2)\,\delta t; \quad \delta u = [C + (n-1)(\alpha_1 + \alpha_2 x)]\,u\delta t.$

Our group therefore has the following infinitesimal transformations:

$$\frac{\partial f}{\partial x}, \quad x\frac{\partial f}{\partial x}, \quad u\frac{\partial f}{\partial u}, \quad x^2\frac{\partial f}{\partial x} + (n-1)\,xu\,\frac{\partial f}{\partial u}. \qquad (45)$$

The infinite group (11) for this case reduces to a finite group with 4 parameters. The finite equations of this group are

$$x_1 = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad u_1 = \frac{u}{(\gamma x + \delta)^{n-1}}. \qquad (46)$$

Equation (35) now reduces to

$$\delta u^{(k)} = \left[\left(C + \frac{n-2k-1}{2}\,\xi'\right)u^{(k)} + \frac{k(n-k)}{2}\,u^{(k-1)}\right]\delta t, \qquad (47)$$

and (37) becomes

$$\delta q_j^{(k)} = \{-(k+j)\,q_j^{(k)}\zeta' - \tfrac{1}{2}\,[k(k+2j-1)\,q_j^{(k-1)} + j(j-1)\,q_{j-1}^{(k)}]\,\xi''\}\,\delta t. \qquad (48)$$

$$(j = 3, 4, \ldots, n; \;\; k = 0, 1, 2, \ldots; \;\; q_2 = 0)$$

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* Comptes Rendus, vol. 88 (1879), p. 226; see also Forsyth, Phil. Trans., 1888, I, p. 403.

§43. We find then

Equations to be satisfied by invariants of (43) *involving only* $q_j^{(k)}$, *where* $k+j \leqq w$.

$$\left.\begin{aligned} 0 &= A_1^{(w)}f \equiv \sum_{3}^{\left\{\substack{w\\ n}\right.} {}_j \sum_{0}^{w-j} {}_k (k+j)\, q_j^{(k)} \frac{\partial f}{\partial q_j^{(k)}}\,. \\ 0 &= A_2^{(w)}f \equiv \sum_{3}^{\left\{\substack{w-1\\ n}\right.} {}_j \sum_{1}^{w-1} {}_k\, k(k+2j-1)\, q_j^{(k-1)} \frac{\partial f}{\partial q_j^{(k)}} + \sum_{4}^{\left\{\substack{w\\ n}\right.} {}_j \sum_{0}^{w-j} {}_k\, j(j-1)\, q_{j-1}^{(k)} \frac{\partial f}{\partial q_j^{(k)}}. \end{aligned}\right\} \quad (49)$$

Equations to be satisfied by invariants of (43) *involving* $q_j^{(k)}$, $(k+j \leqq w)$, *and* $u^{(l)}\,(l \leqq w)$.

$$\left.\begin{aligned} 0 &= X_0^{(w)}f \equiv \sum_{0}^{\left\{\substack{n-1\\ w}\right.} {}_k\, u^{(k)} \frac{\partial f}{\partial u^{(k)}}\,, \\ 0 &= X_1^{(w)}f \equiv \sum_{1}^{\left\{\substack{n-1\\ w}\right.} {}_k\, k u^{(k)} \frac{\partial f}{\partial u^{(k)}} + \sum_{3}^{\left\{\substack{w\\ n}\right.} {}_j \sum_{0}^{w-j} {}_k (k+j)\, q_j^{(k)} \frac{\partial f}{\partial q_j^{(k)}}\,, \\ 0 &= X_2^{(w)}f \equiv \sum_{1}^{\left\{\substack{n-1\\ w}\right.} {}_k\, k(n-k)\, u^{(k-1)} \frac{\partial f}{\partial u^{(k)}} - A_2^{(w)}f, \end{aligned}\right\} \quad (50)$$

where $A_2^{(w)}f$ is given by (49).

§44. The number of solutions of (49) when $w \leqq n$ is $\frac{1}{2}(w-2)(w-1)-2 = \frac{1}{2}(w^2-3w-2)$, for the equations are independent. When $w > n$, the number of solutions is $(n-2)w - \frac{1}{2}(n^2-n+2)$. The number of solutions of (50) when $w < n$ is $\frac{1}{2}(w-2)(w+1)$, and when $w \geqq n$ is $(n-2)w - \frac{1}{2}(n^2-3n+4)$.

The number of solutions of (50) which must contain $u^{(l)}$ is w or $n-1$, according as $w < n$ or $w \geqq n$. All except *two* of these solutions may be chosen free from $q^{(k)}$. They are therefore the same for *all* equations (43) of the n^{th} order. Hence *a linear differential equation of the form* (43) *has two, and only two, characteristic covariants.*

By writing out the first few terms of (50) it is easily proved that these two covariants may be chosen as

$$\frac{6q_3u' + (n-1)\,q_3'u}{uq_3^{\frac{4}{3}}} \quad \text{and} \quad \frac{(n-1)\,u''u - (n-2)\,u'^2}{u^2q_3^{\frac{2}{3}}}. \qquad (51)$$

Complete Solution of the System (50).

§45. This problem may be divided into two parts, viz.

1. The problem of finding all the solutions of (49).
2. The problem of finding all those solutions of (50) which contain $u^{(l)}$ only.

These two classes of solutions, together with the two solutions (51), constitute the complete solution of (50).

§46. Let us first find all those solutions of (50) which contain $u^{(l)}$ only. These are called by Forsyth* the "identical covariants." They are the solutions of

$$\sum_{0}^{\left\{\substack{n-1\\ w}\right.}{}^{k}\, u^{(k)}\frac{\partial f}{\partial u^{(k)}}=0,\quad \sum_{1}^{\left\{\substack{n-1\\ w}\right.}{}^{k}\, ku^{(k)}\frac{\partial f}{\partial u^{(k)}}=0,\quad \sum_{1}^{\left\{\substack{n-1\\ w}\right.}{}^{k}\, k(n-k)\,u^{(k-1)}\frac{\partial f}{\partial u^{(k)}}=0. \quad (52)$$

We can easily find all the solutions of these equations by making use of the general principles given in §§18, 19. The group (46) is evidently a subgroup of (11) for which

$$\Psi=C\chi^{\frac{n-1}{2}},$$

and therefore if we can find one relative invariant the rest may be found by using (22). The matrix of the three equations (52) is

$$\left\|\begin{matrix} u & u' & u'' & u''' & \dots \\ 0 & u' & 2u'' & 3u''' & \dots \\ 0 & (n-1)\,u & 2\,(n-2)\,u' & 3\,(n-3)\,u'' & \dots \end{matrix}\right\|.$$

Equating the first three-rowed determinant to zero gives

$$U_2=(n-1)\,u''u-(n-2)\,u'^2$$

as a relative invariant, and this is the *only* relative invariant which can be found by forming determinants. Now using (22) gives

$$\frac{\frac{n-1}{2}\,u\,U_2'-\left(2\,\frac{n-1}{2}-2\right)u'\,U_2}{U_2^{\frac{3}{2}}},$$

or
$$\Phi_3=\frac{(n-1)^2\,u'''\,u^2-3\,(n-1)(n-3)\,u''\,u'\,u+2\,(n-2)(n-3)\,u'^3}{U_2^{\frac{3}{2}}}$$

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* Phil. Trans., 1888, I, p. 428.

as an absolute invariant. Then by (21)

$$\Phi_4 = \left(\frac{u^2}{U_2}\right)^{\frac{1}{2}} \frac{d\Phi_3}{dx} = \Delta\Phi_3$$

is a second absolute invariant, and continuing the process we find

$$\Phi_3, \ \Delta\Phi_3, \ \Delta^2\Phi_3, \ldots\ldots, \Delta^{w-3}\Phi_3,$$

as all the solutions of (50) involving only u and its derivatives, the last one only involving $u^{(w)}$. These solutions are evidently independent. This method does not seem to readily lend itself to the computation of all the solutions in explicit form. We therefore proceed in the following manner:

§47. Let us assume at first that $w \leqq n-1$. The first of the equations (52) tells us that f is homogeneous of degree zero in $u^{(i)}$, and the second that f is isobaric of weight zero in $u^{(i)}$. u is a solution of the second and third equations, and $U_2 = (n-1)\,u''u - (n-2)\,u'^2$ is a solution of the third. If then we can find an isobaric and homogeneous solution of the third equation, we can make it isobaric of weight zero by dividing by a suitable power of U_2, which is homogeneous, and isobaric of weight 2. We can then make the function homogeneous of degree zero by multiplying by a power of u. The result will be a solution of (52). (This process amounts to finding another relative invariant, and applying (21)). We have then to find a homogeneous and isobaric solution of the third equation of (52). There are evidently no solutions *linear* in $u^{(i)}$. There are, however, isobaric solutions which are homogeneous of the second degree in $u^{(i)}$. When $2j \leqq w$, and we write

$$F_{2j} = \sum_{0}^{j-1}{}_{i}\, \beta_i u^{(2j-i)} u^{(i)} + \tfrac{1}{2}\beta_j\, [u^{(j)}]^2,$$

where β_i is an undetermined constant, we have

$$\begin{aligned} X_3^{(w)} F_{2j} &= \sum_{1}^{j}{}_{k}\, k\,(n-k)\,\beta_k u^{(k-1)} u^{(2j-k)} + \sum_{j+1}^{2j}{}_{k}\, k\,(n-k)\,\beta_{2j-k} u^{(k-1)} n^{(2j-k)} \\ &= \sum_{1}^{j}{}_{k}\, [k\,(n-k)\,\beta_k + (2j-k+1)(n-2j+k-1)\,\beta_{k-1}]\, u^{(k-1)} u^{(2j-k)}. \end{aligned}$$

If now

$$\beta_k = -\frac{(2j-k+1)(n-2j+k-1)}{k\,(n-k)}\,\beta_{k-1},$$

or

$$\beta_k = (-1)^k \frac{(2j)!\ (n-2j+k-1)!\ (n-k-1)!}{(2j-k)!\ k!\ \ (n-2j-1)!\ (n-1)!}, \qquad (53)$$
$$(k = 0,\ 1,\ 2,\ \ldots\ldots,\ j)$$

then we have

$$U_{2j} = \overset{j-1}{\underset{0}{\sum\nolimits_k}} \beta_k u^{(2j-k)} u^{(k)} + \tfrac{1}{2}\beta_j \left[u^{(j)}\right]^2, \qquad (54)$$
$$\left(j = 1,\ 2,\ \ldots\ldots \frac{w}{2} \text{ or } \frac{w-1}{2}\right)$$

is a solution of $X_2^{(w)} f = 0$. U_{2j} is isobaric of weight $2j$, and homogeneous of the second degree.

$$\Phi_{2j} = \frac{u^{2j-3} U_{2j}}{U_2^j}, \qquad (55)$$
$$\left(j = 2,\ 3,\ \ldots\ldots \frac{w}{2} \text{ or } \frac{w-1}{2}\right)$$

is therefore a solution of (52). Since $2j \leq w$, the number of such solutions is $\frac{w}{2} - 1$ or $\frac{w-3}{2}$, according as w is even or odd. There are in all $w-2$ solutions involving $u^{(l)}$ alone. Hence we have still to find $\frac{w}{2} - 1$ or $\frac{w-1}{2}$ solutions of (52).

§48. When we attempt to apply the foregoing method to find solutions of $X_2^{(w)} f = 0$ which are isobaric of *odd* weight and homogeneous of the second degree, we find that all the coefficients are zero, and there are no such solutions. The next step would be to assume as solution an isobaric function homogeneous of the third degree, but the work of substituting such a function in the equation $X_2^{(w)} f = 0$ would be long. We might make use of the general method and choose solutions of the form

$$\frac{d\Phi_{2j}}{dx} \bigg/ \frac{d\Phi_3}{dx}, \qquad \left(j = 2,\ 3,\ \ldots\ldots \frac{w}{2} - 1, \text{ or } \frac{w-1}{2}\right)$$

where Φ_3 has the value given in §46. This would give us a number of solutions which are evidently independent of those already found, and exactly equal in number to those we still need. They would therefore complete the solution. If, however, we take the solutions of this form, they are unnecessarily complicated. The numerator of the fraction would be u^{2j-3} multiplied into a function homogeneous of the *fifth* degree and isobaric of weight $2j+1$. This function is a solution of $X_2^{(w)}f = 0$, but it is not the simplest solution which is of weight $2j+1$. We shall now prove that there are such solutions homogeneous of the *third* degree.

Theorem III applied to (55) shows that U_{2j} is a relative invariant. Then (22) shows that an absolute invariant is

$$\frac{(n-1)\,u\,U'_{2j} - 2\,(n-2j-1)\,u'\,U_{2j}}{u^{\frac{j-1}{j}}\;U_{2j}^{\frac{2j+1}{2j}}}.$$

When the value of U_{2j} from (54) is substituted here we get a solution of (50). If we write

$$U_{2j+1} = (n-1)\,u\,U'_{2j} - 2\,(n-2j-1)\,u'\,U_{2j}, \tag{56}$$

then

$$\Phi_{2j+1} = \frac{u^{2j-3}\,U_{2j+1}}{U_3^{\frac{2j+1}{3}}}, \qquad \left(j = 1,\ 2,\ \ldots,\ \frac{w}{2} - 1 \text{ or } \frac{w-1}{2}\right) \tag{57}$$

is the solution of (50) which we sought. U_{2j+1} is isobaric of weight $2j+1$ and homogeneous of the third degree. (57) gives $\frac{w}{2} - 1$ or $\frac{w-1}{2}$ solutions according as w is even or odd, and these, together with the solutions Φ_{2j}, $\frac{w}{2} - 1$ or $\frac{w-3}{2}$ in number, are the $w-2$ independent solutions which involve $u^{(i)}$ alone.

The treatment of the case $w \leqq n$ is exactly the same as in the general case, and will not be further considered.

§49. Having now found all the solutions of (50) which necessarily involve u or its derivatives, we must next find all the solutions of (49). The equation $A_1^{(w)}f = 0$ simply shows that the solution must be isobaric of weight zero in the $q_j^{(k)}$'s. q_3 is a solution of the second equation. Therefore if we can find any other isobaric sulution of the second equation, we can make it isobaric of weight zero by dividing it by a suitable power of q_3 and thus get a solution of the system. Let us first assume $w \leqq n$. Then we may write $A_2^{(w)}f = 0$ in the form

$$0 = A_2^{(w)}f \equiv \sum_4^w \sum_3^{i-1} \left[(i-j)(i+j-1)\frac{\partial f}{\partial q_j^{(i-j)}} + j\,(j+1)\frac{\partial f}{\partial q_{j+1}^{(i-j-1)}}\right] q_3^{(i-j-1)}.$$

Let us see whether any solutions of the first degree in $q_j^{(k)}$ and isobaric of weight m exist. Assume

$$\Theta_m = \sum_0^{m-3} \alpha_{m,s}\, q_{m-s}^{(s)}. \qquad (4 \leqq m \leqq w)$$

Then

$$A_3^{(w)}\Theta_m = \sum_1^{m-1} \left[(m-j)(m+j-1)\,\alpha_{m,\,m-j} + j(j+1)\,\alpha_{m,\,m-j-1}\right] q_j^{(m-j-1)}.$$

Θ_m is therefore a solution of $A_3^{(w)}f = 0$ if

$$\alpha_{m,\,m-j} = -\frac{j(j+1)}{(m-j)(m+j-1)}\,\alpha_{m,\,m-j-1}, \qquad (j = 3, 4, \ldots, m-1)$$

$$\alpha_{m,\,s} = -\frac{(m-s)(m-s+1)}{s\,(2m-s-1)}\,\alpha_{m,\,s-1}, \qquad (s = 1, 2, \ldots, m-3)$$

$$= (-1)^s \frac{(m-2)!\ m!\ (2m-s-2)!}{(m-s-1)!\ (m-s)!\ (2m-3)!\ s!}\cdot\frac{\alpha_{m,\,0}}{2}.$$

The last equation is an identity for $s = 0$. If we choose $\alpha_{m,\,0} = 2$, then

$$\alpha_{m,\,s} = (-1)^s \frac{m!\ (m-2)!\ (2m-s-2)!}{(m-s)!\ s!\ (m-s-1)!\ (2m-3)!}, \qquad (58)$$

$$(s = 0, 1, 2, \ldots, m-3)$$

Therefore with this value of $\alpha_{m,\,s}$

$$\Theta_m = \sum_0^{m-3} \alpha_{m,\,s}\, q_{m-s}^{(s)}, \qquad (m = 4, 5, \ldots, w) \qquad (59)$$

is a solution of $A_3^{(w)}f = 0$. Then

$$J = \frac{\Theta_m}{q_3^{\frac{m}{3}}}, \qquad (m = 4, 5, \ldots, w) \qquad (60)$$

is a solution of (49). We have $w - 3$ such solutions. Direct integration of (49) shows that

$$J_3 = \frac{6q_3''\,q_3 - 7q_3'}{q_3^{\frac{8}{3}}} \qquad (61)$$

is a solution. From these $w - 2$ solutions all the others,

$$\tfrac{1}{2}(w^2 - 3w - 2) - (w - 2) = \tfrac{1}{2}(w^2 - 5w + 2)$$

in number, can be found by differentiation processes. We have simply to apply (21). If we define the operator ϑ by

$$\vartheta = \frac{1}{q_3^{\frac{1}{3}}}\,\frac{d}{dx}, \qquad (62)$$

then (21) gives us as absolute invariants (and hence solutions of (49)) the following set:

$$\left.\begin{array}{l} J_m,\ \vartheta J_m,\ \vartheta^2 J_m,\ \ldots\ldots,\ \vartheta^{w-m}J_m;\quad (m=4,5,\ldots\ldots,w) \\ J_3,\ \vartheta J_3,\ \vartheta^2 J_3,\ \ldots\ldots,\ \vartheta^{w-5}J_3. \end{array}\right\} \tag{63}$$

These solutions are independent of one another, and their number is

$$(w-4)+[(w-3)+(w-4)+(w-5)+\ldots\ldots+2+1]=\tfrac{1}{2}(w^2-3w-2);$$

that is, (63) gives all the independent solutions of (49) for $w \leqq n$. If $w > n$, further differentiation of $I_3, \ldots\ldots, I_n$ is seen to give all the solutions.

§50. *To summarize, the complete solution of* (50) *consists of the following invariants:*

$$\frac{6q_3u'+(n-1)q_3'u}{q_3^{\frac{4}{3}}u} \text{ and } \frac{(n-1)u''u-(n-2)u'^2}{q_3^{\frac{2}{3}}u^2}, \tag{51}$$

$$\Phi_{2j}=\frac{u^{2j-2}U_{2j}}{U_2^{j}}, \qquad \left(j=2,3,\ldots\ldots,\frac{w}{2} \text{ or } \frac{w-1}{2}\right) \tag{55}$$

$$\Phi_{2j+1}=\frac{u^{2j-2}U_{2j+1}}{U_3^{\frac{2j+1}{3}}}, \quad \left(j=1,2,\ldots\ldots,\frac{w}{2}-1 \text{ or } \frac{w-1}{2}\right) \tag{57}$$

$$J_3=\frac{6q_3''q_3-7q_3'^2}{q_3^{\frac{8}{3}}}, \tag{61}$$

$$J_m=\frac{\Theta_m}{q_3^{\frac{m}{3}}}, \qquad (m=4,5,\ldots\ldots;w) \tag{60}$$

$$\left.\begin{array}{l} \vartheta J_m,\ \vartheta^2 J_m,\ \ldots\ldots,\ \vartheta^{w-m}J_m, \\ \vartheta J_3,\ \vartheta J_3\ ,\ \ldots\ldots,\ \vartheta^{w-5}J_3\ . \end{array}\right\}(m=4,5,\ldots\ldots,w) \tag{63}$$

In these formulæ the quantities U_{2j}, U_{2j+1}, Θ_m and ϑ are defined by (54), (56), (59) and (62) respectively. These formulæ give the $\frac{1}{2}(w^2-3w-2)$ invariants and the w covariants (which involve no letter of weight greater than $w \leqq n$) of the equation in canonical form

$$u^{(n)}+\sum_3^n \frac{n!}{(n-s)!\ s!}q_s u^{(n-s)}=0 \tag{43}$$

for all transformations which leave this equation invariant, namely, for the 4-parameter group

$$x_1=\frac{\alpha x+\beta}{\gamma x+\delta}, \quad u_1=\frac{u}{(\gamma x+\delta)^{n-1}}. \tag{46}$$

Of these invariants, U_{2j}, U_{2j+1} and J_m are exactly the same as those given by Forsyth. The latter, however, gives instead of the derived set (63) another set of invariants, which of course must be functionally equivalent to (60), (61) and (63).

The Differential Parameter.

§51. In conclusion we shall show that all the different methods of which we have made use in deriving new invariants from known invariants by differentiation, are only special cases of what Lie calls the "Differential Parameter."*

Consider first the case of Theorem IV, where two invariants, I_1 and I_2 are known, and $\dfrac{dI_2}{dx}\Big/\dfrac{dI_1}{dx}$ is shown to be invariant. The differential parameter is a function

$$\Omega\,(x,\, y,\, y',\, \ldots\ldots,\, a^{(k)} \ldots\ldots I_1,\, I_2,\, I_1',\, I_2' \ldots\ldots)$$

such that when I_1 and I_2 are invariants, Ω also is invariant. We have

$$\begin{aligned}\delta\Omega = \frac{\partial\Omega}{\partial x}\,\delta x + \sum_0^w{}_k \frac{\partial\Omega}{\partial y^{(k)}}\,\delta y^{(k)} + \sum_1^{\left\{{w \atop n}\right.}{}_j \sum_0^{w-j}{}_k \frac{\partial\Omega}{\partial a_j^{(k)}}\,\delta a_j^{(k)} + \frac{\partial\Omega}{\partial I_1}\,\delta I_1 + \frac{\partial\Omega}{\partial I_2}\,\delta I_2 \\ + \frac{\partial\Omega}{\partial I_1'}\,\delta I_1' + \frac{\partial\Omega}{\partial I_2'}\,\delta I_2' + \ldots\ldots \\ = X^{(w\,w)}\Omega\,.\,\delta t + \frac{\partial\Omega}{\partial I_1}\,\delta I_1 + \frac{\partial\Omega}{\partial I_2}\,\delta I_2 + \frac{\partial\Omega}{\partial I_1'}\,\delta I_1' + \frac{\partial\Omega}{\partial I_2'}\,\delta I_2' + \ldots\ldots,\end{aligned}$$

if we make use of the notation of (28). Moreover, we have

$$\delta I_1' = \delta\,\frac{dI_1}{dx} = \frac{d}{dx}\,\delta I_1 - I_1'\,\frac{d\delta x}{dx} = -\,I_1'\,\xi'\,\delta t, \text{ since } \delta I_1 = 0.$$

Since Ω is to be invariant when I_1 and I_2 are, we have $\delta\Omega = 0$ when $\delta I_1 = 0$, $\delta I_2 = 0$, and this for all values of ϕ and ξ. Hence, on substituting the values of $\delta I_1'$ and $\delta I_2'$, using (30) and (31) and equating the coefficients of $\phi^{(s)}$ and $\xi^{(s)}$ to zero, we get

$$\begin{aligned}0 &= Y_s\Omega, &&(s = 0,\, 1,\, \ldots\ldots,\, w)\\ 0 &= X_{s+1}\Omega, &&(s = 1,\, 2,\, \ldots\ldots,\, w)\\ 0 &= -X_1\Omega + I_1'\,\frac{\partial\Omega}{\partial I_1'} + I_2'\,\frac{\partial\Omega}{\partial I_2'}\,.\end{aligned}$$

This is a complete system of equations giving Ω. The system has all the solutions of (30) and two others, which are the differential parameters. One is evidently

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*Lie, "Continuierliche Gruppen," herausgegeben von Scheffers, pp. 670, 680, 789.

$$\Omega_1 = \frac{I_2'}{I_1'} = \frac{dI_2}{dx}\bigg/\frac{dI_1}{dx},$$

which is the result given in the corollary to Theorem IV. The other solution is

$$\Omega_2 = (2a_3 - 3a_2' - 6a_2a_1 + 4a_1^3 + 6a_1a_1' + a_1'')^{-\frac{1}{3}} I_1'.$$

No use has been made of this second solution in the earlier part of the foregoing paper, but in Chapter 5 it is used in formula (62).

Next take the case of §18, where two relative invariants, U and V, are known, each of degree λ and weight w. The ratio $I = U/V$ is then a known invariant. As before, let

$$\Omega(x, \ldots\ldots, y^{(i)}, \ldots\ldots, a_j^{(k)}, \ldots\ldots, U, V, I, I')$$

be the differential parameter. Then

$$\delta\Omega = X^{(ww)}\Omega \,.\, \delta t + \frac{\partial\Omega}{\partial U}\delta U + \frac{\partial\Omega}{\partial V}\delta V + \frac{\partial\Omega}{\partial I}\delta I + \frac{\partial\Omega}{\partial I_1'}\delta I'.$$

Considerations similar to those of §39 show that U and V are solutions of

$$\begin{aligned} 0 &= Y_s f, && (s = 1, 2, \ldots\ldots, w) \\ 0 &= X_{s+1} f, && (s = 1, 2, \ldots\ldots, w) \\ \lambda f &= Y_0 f, && \\ wf &= -X_1 f, && \end{aligned}$$

$$\therefore\ \delta U = X^{(ww)} U \,.\, \delta t = (\phi Y_0 U + \xi' X_1 U)\,\delta t = (\lambda\phi - w\xi')\, U \,.\, \delta t,$$
$$\delta V = (\lambda\phi - w\xi')\, V \,.\, \delta t.$$

As before $\delta I' = -I'\xi'\,\delta t$. Then $\delta\Omega = 0$, $\delta I = 0$ gives

$$0 = \xi\frac{\partial\Omega}{\partial x} + \sum_0^w \phi^{(s)} Y_s\Omega + \sum_1^{w+1} \xi^{(s)} X_s\Omega + (\lambda\phi - w\xi')\left(U\frac{\partial\Omega}{\partial U} + V\frac{\partial\Omega}{\partial V}\right) - \xi' I'\frac{\partial\Omega}{\partial I'}.$$

This breaks up into

$$\begin{aligned} 0 &= Y_s\Omega, && (s = 1, 2, \ldots\ldots, w) \\ 0 &= X_{s+1}\Omega, && (s = 1, 2, \ldots\ldots, w) \\ 0 &= Y_0\Omega + \lambda\left(U\frac{\partial\Omega}{\partial U} + V\frac{\partial\Omega}{\partial V}\right), && \\ 0 &= X_1\Omega - w\left(U\frac{\partial\Omega}{\partial U} + V\frac{\partial\Omega}{\partial V}\right) - I'\frac{\partial\Omega}{\partial I'}. && \end{aligned}$$

These equations have three more solutions than (30). One of them is U/V, the second is Ω_2 given earlier, and the third is the desired differential parameter, viz.

$$\Omega_3 = \left[\frac{y^\lambda}{V}\right]^{\frac{1}{w}} I',$$

This is the same as (21). It should be noticed that no use has been made of the fact that $I = U/V$, so that in the last formula I may be any invariant whatever and V any relative invariant of degree λ and weight w.

Lastly, consider the case of §19, and suppose that we know a single relative invariant, U, of degree λ and weight w. For this case $\phi = \gamma + \nu\xi'$, where γ is an arbitrary constant. The transformation becomes

$$X^{(ww)}f = \xi\frac{\partial f}{\partial x} + \gamma Y_0 f + \sum_{1}^{w+1}(\nu Y_{s-1}f + X_s f)\,\xi^{(s)}.$$

Let us see whether there is a differential parameter of the form

$$\Omega(x, \ldots\ldots, y^{(l)}, \ldots\ldots a_j^{(k)}, \ldots\ldots, U, U')\text{-}$$

We have, as before,

$$\begin{aligned}
\delta U &= (\lambda\phi - w\xi')\,U\delta t = [\lambda\gamma + (\lambda\nu - w)\,\xi']\,U\delta t,\\
\delta U' &= \frac{d}{dx}\delta U - U'\frac{d\delta x}{dx}\\
&= (\lambda\nu - w)\,\xi'' U\delta t + [\lambda\gamma + (\lambda\nu - w - 1)\,\xi']\,U'.\delta t.
\end{aligned}$$

Substituting the values of δU and $\delta U'$, and equating the coefficients of γ and $\xi^{(s)}$ to zero gives

$$\begin{aligned}
0 &= Y_0\Omega + \lambda U\frac{\partial\Omega}{\partial U} + \lambda U'\frac{\partial\Omega}{\partial U'},\\
0 &= X_1\Omega - wU\frac{\partial\Omega}{\partial U} - (w+1)\,U'\frac{\partial\Omega}{\partial U'},\\
0 &= \nu Y_1\Omega + X_2\Omega + (\lambda\nu - w)\,U\frac{\partial\Omega}{\partial U'},\\
0 &= \nu Y_{s-1}\Omega + X_s\Omega. \qquad (s = 3, 4, \ldots\ldots, w+1)
\end{aligned}$$

Expanding by using (30) gives

$$\begin{aligned}
0 &= y\frac{\partial\Omega}{\partial y} + y'\frac{\partial\Omega}{\partial y'} + \lambda U\frac{\partial\Omega}{\partial U} + \lambda U'\frac{\partial\Omega}{\partial U'} + \ldots\ldots,\\
0 &= y'\frac{\partial\Omega}{\partial y'} + wU\frac{\partial\Omega}{\partial U} + (w+1)\,U'\frac{\partial\Omega}{\partial U'} + \ldots\ldots,\\
0 &= \nu y\frac{\partial\Omega}{\partial y'} + (\lambda\nu - w)\,U\frac{\partial\Omega}{\partial U'} + \ldots\ldots,
\end{aligned}$$

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where the dots indicate terms which do not involve derivatives with regard to y, y', U or U'. Neglecting these terms we get a complete system with the one solution

$$\Omega_4 = \frac{(\lambda v - w)\, U' y - v\, U y'}{y^{\frac{w-\lambda}{w}}\, U^{\frac{w+1}{w}}} \,.$$

This is the same as (22). In computing the first two differential parameters, Ω_1 and Ω_3, no use whatever was made of the finite transformations of the group. For the last two, however, use was made of the fact that the factor of (17) is known.

On Systems of Multiform Functions belonging to a Group of Linear Substitutions with Uniform Coefficients.

BY E. J. WILCZYNSKI.

INTRODUCTION.

The integrals of a linear differential equation with uniform coefficients have the characteristic property of being uniform and continuous everywhere except in the vicinity of the singular points of the equation, where they undergo, in general, linear substitutions with constant coefficients.

While Fuchs takes the differential equation to be given, and the essential problem for him is the determination of the substitution group belonging to its integrals, Riemann takes up the converse problem. He supposes the branch-points and the fundamental substitutions to be given arbitrarily. The question then arises as to the existence of a system of functions having the given substitutions and branch-points. In the cases treated by him, Riemann proves the existence theorem by passing to the differential equation and finding the arithmetical expressions for the functions. A direct proof, similar to that furnished by Schwarz and Neumann as basis for Riemann's theory of Abelian Functions, has been given by Klein for a very special case only.*

It will therefore not be surprising that we encounter still greater difficulties if we attempt to prove the existence of the general functions, studied in this paper, as I believe, for the first time. Nothing like a general and direct existence theorem is therefore to be found in it. But the existence of a large and important class of these functions is demonstrated by an indirect method, which consists essentially in generalizing the hypergeometric functions in a proper manner.

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* Klein in Ritter's memoir, Math. Ann., Bd. 41.

The first part of the paper treats of the theory of the functions, so far as concerns their behavior in the vicinity of any singular point. This suffices to establish the existence theorem in some very simple special cases. The latter part of the paper deals with the generalized hypergeometric functions.

§1.—*Formulation of the Problem.*

Let $(y_1, y_2, \ldots, y_n)$ be a system of n functions of x, which are uniform everywhere except in the vicinity of m points, called branch-points, $a_1, a_2, \ldots, a_m$. Suppose moreover that when x makes a circuit around a_i in the positive direction, the system $(y_1, y_2, \ldots, y_n)$ undergoes a linear substitution with *variable* coefficients. It is therefore supposed that the coefficients of the substitutions are themselves functions of x, and, as we shall mostly assume, *uniform* functions. But we shall also to some extent consider the case that these coefficients are multiform with a finite number of branches.

We will assume that the n functions $(y_1, \ldots, y_n)$ are *independent*, signifying thereby that no relation of the form

$$\psi_1 y_1 + \psi_2 y_2 + \ldots + \psi_n y_n = 0$$

can be identically verified, where $\psi_1, \psi_2, \ldots, \psi_n$ are uniform functions of x in the vicinity of the singular points a_i.

Let

$$A_i = \begin{pmatrix} \alpha_{11}^{(i)}, & \alpha_{12}^{(i)}, & \ldots, & \alpha_{1n}^{(i)} \\ \alpha_{21}^{(i)}, & \alpha_{22}^{(i)}, & \ldots, & \alpha_{2n}^{(i)} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{n1}^{(i)}, & \alpha_{n2}^{(i)}, & \ldots, & \alpha_{nn}^{(i)} \end{pmatrix}, \quad (i = 1, 2, \ldots, m) \qquad (1)$$

be the substitution which $(y_1, \ldots, y_n)$ undergoes when x describes a positive circuit around a_i.

Obviously the relation must hold

$$A_1 A_2 \ldots A_m = 1 \qquad (2)$$

if 1 denotes identity. This suffices to show that no such systems of functions, which for briefness we will call Λ functions, are possible with only one branch-point. If there are two branch-points the two substitutions are inverses of each other.

It is the purpose of this paper to develop the general properties of the Λ

functions, and to establish their existence in some general cases. The analogy to the theory of linear differential equations is striking, and most of our results will be found by methods familiar to those who are interested in that theory.

§2.—*The Characteristic Equation.*

Let a be any one of the branch-points, and A the corresponding substitution. If we put

$$z = \lambda_1 y_1 + \lambda_2 y_2 + \ldots . + \lambda_n y_n , \tag{3}$$

$\lambda_1, \lambda_2, \ldots . , \lambda_n$ can generally be so determined as functions of x, uniform in the vicinity of $x = a$, that after a circuit around $x = a$, *z will change into $\bar{z} = \omega z$, where ω is also uniform in the vicinity of $x = a$.*

In order that this may be so, the equations

$$\left.\begin{array}{l} \lambda_1 (\alpha_{11} - \omega) + \lambda_2 \alpha_{21} + \ldots . + \lambda_n \alpha_{n1} \qquad = 0 , \\ \lambda_1 \ \alpha_{12} + \lambda_2 (\alpha_{22} - \omega + \ldots . + \lambda_n \alpha_{n2} \qquad = 0 , \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \lambda_1 \ \alpha_{1n} + \lambda_2 \ \alpha_{2n} \qquad + \ldots . + \lambda_n (\alpha_{nn} - \omega) = 0 \end{array}\right\} \tag{4}$$

must be verified, since $y_1, \ldots . , y_n$ are supposed to be independent. Therefore, the case $\lambda_1 = \lambda_2 = \ldots . = \lambda_n = 0$ being of course excluded, we must have

$$F(\omega) = \begin{vmatrix} \alpha_{11} - \omega , & \alpha_{21} & , \ldots . , \alpha_{n1} \\ \alpha_{12} & , \ \alpha_{22} - \omega , & \ldots . , \alpha_{n2} \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ \alpha_{1n} & , \ \alpha_{2n} & , \ldots . , \alpha_{nn} - \omega \end{vmatrix} = 0 , \tag{5}$$

which is the *characteristic equation.* The above theorem will then, as equations (3), (4), (5) show, be true if $x = a$ is not a branch-point of the characteristic equation.

In general this equation will have n unequal roots, ω_i. *If $x = a$ is no branch-point for any of them, there will be n linear homogeneous combinations of $(y_1, \ldots . , y_n)$, which we will call $(z_1, \ldots . , z_n)$ such that*

$$\bar{z}_i = \omega_i z_i , \qquad (i = 1, 2, \ldots . , n) \tag{6}$$

denoting always by a dash over a quantity the value which it assumes after a positive circuit around $x = a$.

The substitution expressed by (6) is said to be in the *canonical* form, and the system $(z_1, \ldots . , z_n)$ is to be called the canonical system belonging to $x = a$.

It is important to distinguish whether $F(\omega)$ is reducible or not. If it is irreducible the n roots $\omega_1, \ldots\ldots, \omega_n$ corresponding to each value of x are but the n branches of a single monogenic function of x, so that any one root ω_i can be analytically continued into every other root ω_k, or, as we may also say, from every point of the Riemann's surface belonging to (5) there is a path to every other point of that surface. The multipliers ω_i will then be permutated if x describes closed circuits in that Riemann's surface. If $F(\omega) = 0$ is composed of several irreducible factors, only the roots belonging to each factor are thus interchangeable. In particular, it may happen that $F(\omega)$ is a product of n linear factors. Then $\omega_1, \ldots\ldots, \omega_n$ are absolutely uniform functions of x.

Let us now assume that $x = a$ is a branch-point of the characteristic equation, where the μ roots $\omega_1, \omega_2, \ldots\ldots, \omega_\mu$ are interchanged cyclically, so that

$$\omega_1 = \omega_2, \ \bar{\omega}_2 = \omega_3, \ \ldots\ldots, \ \bar{\omega}_{\mu-1} = \omega_\mu, \ \bar{\omega}_\mu = \omega_1.$$

Let us moreover denote the λ's which result from (4), when ω is put equal to ω_i, by λ_{ik} $(k = 1, 2, \ldots\ldots, n)$ and the corresponding z resulting from (3) by z_i. *Then obviously if we determine* $z_1, \ldots\ldots, z_\mu$ *by the same equations* (3), (4), (5) *which hold in the general case,*

$$\bar{z}_1 = \omega_2 z_2, \ \bar{z}_2 = \omega_3 z_3, \ \ldots\ldots, \ \bar{z}_{\mu-1} = \omega_\mu z_\mu, \ \bar{z}_\mu = \omega_1 z_1, \tag{7}$$

and similarly for any group of roots which has $x = a$ as a branch-point. These equations will in this case be termed the canonical substitutions instead of (6).

If $x = a$ *is no branch-point and* $\omega_1, \ldots\ldots, \omega_n$ *are unequal roots of* $F(\omega) = 0$, $z_1, \ldots\ldots, z_n$ *are independent.* For if there were a relation with coefficients γ_i uniform for $x = a$,

$$\gamma_1 z_1 + \gamma_2 z_2 + \ldots\ldots + \gamma_n z_n = 0,$$

by making circuits around $x = a$, we would find

$$\begin{gathered}\omega_1\gamma_1 z_1 + \omega_2\gamma_2 z_2 + \ldots\ldots + \omega_n\gamma_n z_n = 0; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega_1^{n-1}\gamma_1 z_1 + \omega_2^{n-1}\nu_2 z_2 + \ldots\ldots + \omega_n^{n-1}\gamma_n z_n = 0.\end{gathered}$$

But unless $\gamma_1 = \gamma_2 = \ldots\ldots = \gamma_n = 0$, this is impossible, because the determinant

$$\begin{vmatrix} 1 & , & 1 & , \ldots\ldots, & 1 \\ \omega_1 & , & \omega_2 & , \ldots\ldots, & \omega_n \\ \ldots & & \ldots & \ldots & \ldots \\ \omega_1^{n-1}, & & \omega_2^{n-1}, & \ldots\ldots, & \omega_n^{n-1} \end{vmatrix}$$

$$= (\omega_1 - \omega_2) \ldots\ldots (\omega_1 - \omega_n)(\omega_2 - \omega_3) \ldots\ldots (\omega_2 - \omega_n) \ldots\ldots (\omega_{n-1} - \omega_n)$$

cannot vanish for unequal values of $\omega_1, \ldots\ldots, \omega_n$.

Now let $x = a$ be a branch-point. Obviously after μ circuits around a, $(z_1, \ldots\ldots, z_\mu)$, the canonic functions belonging to the cycle of roots $\omega_1, \ldots\ldots, \omega_\mu$ will have changed into $\omega_1^\mu z_1, \ldots\ldots, \omega_\mu^\mu z_\mu$ exactly as if $\omega_1, \ldots\ldots, \omega_\mu$ were uniform for $x = a$. Suppose that there are still other groups of $\nu, \rho, \ldots\ldots$ roots, the members of each group being permutated when x describes a circuit around $x = a$. Let σ be the least common multiple of μ, ν, ρ, etc. Then from a relation

$$\gamma_1 z_1 + \gamma_2 z_2 + \ldots\ldots + \gamma_n z_n = 0$$

would follow by making σ, 2σ, $\ldots\ldots$, $(n-1)\sigma$ circuits

$$\begin{array}{l} \omega_1^\sigma \gamma_1 z_1 \quad + \omega_2^\sigma \gamma_2 z_2 \quad + \ldots\ldots + \omega_n^\sigma \gamma_n z_n \qquad = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega_1^{(n-1)\sigma} \gamma_1 z_1 + \omega_2^{(n-1)\sigma} \gamma_2 z_2 + \ldots\ldots + \omega_n^{(n-1)\sigma} \gamma_n z_n = 0. \end{array}$$

The determinant of these n equations

$$(\omega_1^\sigma - \omega_2^\sigma) \ldots\ldots (\omega_1^\sigma - \omega_n^\sigma)(\omega_2^\sigma - \omega_3^\sigma) \ldots\ldots (\omega_2^\sigma - \omega_n^\sigma) \ldots\ldots (\omega_{n-1}^\sigma - \omega_n^\sigma)$$

can only vanish if the quotient of two ω's is a σ^{th} root of unity. *Therefore if* $x = a$ *is a branch-point of the characteristic equation, and if the quotient of no two roots of this equation is a root of unity, the system of canonical functions, which we have defined, consists of n independent members.*

The cases of equal roots, or if $x = a$ is a branch-point of roots whose quotients are roots of unity, will be treated hereafter.

§3.—*Invariance of the Characteristic Equation.*

Before we can treat the cases mentioned at the close of the last paragraph, we must notice the following theorem:

If instead of $(y_1, y_2, \ldots\ldots, y_n)$ *we consider the system of functions*

$$\eta_i = \mu_{i1} y_1 + \mu_{i2} y_2 + \ldots\ldots + \mu_{in} y_n, \quad \text{Det}\,(\mu_{ik}) \neq 0,$$
$$(i = 1, 2, \ldots\ldots, n)$$

where μ_{ik} *are functions of* x *which are uniform in the vicinity of* $x = a$, *then the characteristic equation belonging to the* η *functions is the same as that for* $(y_1, y_2, \ldots, y_n)$.

To prove this we can proceed exactly as in the theory of linear differential equations. It is therefore unnecessary to reproduce the demonstration.

§4.—*Form of the Canonical Substitutions when the Characteristic Equation has Equal Roots, or Roots whose Quotient is a Root of Unity.*

Let $x = a$ be no branch-point of the λ roots of $\omega_1, \omega_2, \ldots, \omega_\lambda$, which we will suppose to be equal. Then there will be one function z_1 for which

$$\bar{z}_1 = \omega_1 z_1.$$

Since $y_1, \ldots, y_n$ are supposed to be independent, $z_1, y_2, \ldots, y_n$ will be likewise. Then we will have

$$\begin{cases} \bar{z}_1 = \omega_1 z_1, \\ \bar{y}_2 = \beta_{21} z_1 + \beta_{22} y_2 + \ldots + \beta_{2n} y_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \bar{y}_n = \beta_{n1} z_1 + \beta_{n2} y_2 + \ldots + \beta_{nn} y_n, \end{cases}$$

where β_{ik} and ω_1 are uniform in the vicinity of $x = a$. The corresponding characteristic equation becomes

$$\begin{vmatrix} \omega_1 - \omega, & \beta_{21} & , \ldots, & \beta_{n1} \\ 0 \quad , & \beta_{22} - \omega, & \ldots, & \beta_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ 0 \quad , & \beta_{2n} & , \ldots, & \beta_{nn} - \omega \end{vmatrix} = 0,$$

so that, since $F(\omega) = 0$ has the root ω_1 λ times, according to the theorem of the invariance of the characteristic equation, the equation

$$\begin{vmatrix} \beta_{22} - \omega, & \ldots, & \beta_{n2} \\ \ldots & \ldots & \ldots \\ \beta_{2n} & , \ldots, & \beta_{nn} - \omega \end{vmatrix} = 0,$$

has this root $\lambda - 1$ times. The following system of linear equations

$$\beta_{2i} x_2 + \beta_{3i} x_3 + \ldots + \beta_{ni} x_n = \omega_1 x_i$$
$$(i = 2, 3, \ldots, n)$$

therefore admits of solution, and if we put

$$z_2 = x_2 y_2 + x_3 y_3 + \ldots + x_n y_n,$$

we shall find

$$\bar{z}_2 = z_1(\beta_{21}x_2 + \beta_{31}x_3 + \dots + \beta_{n1}x_n)$$
$$+ \sum_{i=3}^{n} y_i(\beta_{2i}x_2 + \beta_{3i}x_3 + \dots + \beta_{ni}x_n).$$

or

$$\bar{z} = \omega_{21}z_1 + \omega_1 z_2,$$

where ω_{21} is also uniform in the vicinity of $x = a$. If ω_1 is absolutely uniform so is ω_{12}.

Now $z_1, z_2, y_3, \dots, y_n$ can be taken as independent members of a system, and we can continue this process so as to find the following theorem:

If ω_1 is a root of the characteristic equation of multiplicity λ, there exists a group of λ functions $z_1, z_2, \dots, z_\lambda$ which take the following values when x describes a positive circuit around $x = a$:

$$\left.\begin{aligned} \bar{z}_1 &= \omega_1 z_1, \\ \bar{z}_2 &= \omega_{21}z_1 + \omega_1 z_2, \\ &\dots\dots\dots\dots\dots\dots \\ \bar{z}_\lambda &= \omega_{\lambda 1}z_1 + \omega_{\lambda 2}z_2 + \dots + \omega_1 z_\lambda. \end{aligned}\right\} \tag{8}$$

If ω_1 is an absolutely uniform function of x, the same is true of the quantities ω_{α}.

If $x = a$ is a branch-point for the μ interchanging roots $\omega_1, \dots, \omega_\mu$ of the characteristic equation, these are of course unequal. The case of equal roots can then only occur if the irreducible factor $f(\omega)$ of $F(\omega)$, of which ω_1 is a root, occurs more than once. Therefore $\omega_1, \omega_2, \dots, \omega_\mu$ will each occur the same number of times λ if $f(\omega)^\lambda$ is the highest power of $f(\omega)$, which is a factor of $F(\omega)$.

Now for one circuit around $x = a$

$$\bar{z}_1 = \omega_2 z_2, \dots, \bar{z}_{\mu-1} = \omega_\mu z_\mu, \bar{z}_\mu = \omega_1 z_1,$$

and for μ circuits

$$\bar{z}_1 = \omega_1^\mu z_1, \dots, \bar{z}_\mu = \omega_\mu^\mu z_\mu.$$

If we put

$$x - a = t^\mu$$

μ circuits of x around $x = a$ will correspond to one of t around $t = 0$ and $\omega_1, \dots, \omega_\mu$ will be uniform as functions of t in the vicinity of $t = 0$. *Let $\omega_i, \omega_i', \dots, \omega_i^{(\lambda-1)}$ for $i = 1, 2, \dots, \mu$ be the λ roots which are equal to ω_i, then, corresponding to each of these i groups of λ equal roots there are λ functions*

$z_{i1}, \ldots . z_{i\lambda}$ *which after one circuit around* $t = 0$, *or* μ *circuits around* $x = a$, *change into*

$$\left.\begin{aligned} \bar{z}_{i1} &= \omega_i^\mu z_{i1}, \\ \bar{z}_{i2} &= \omega_{21}^{(i)} z_{i1} + \omega_i^\mu z_{i2}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \bar{z}_{i\lambda} &= \omega_{\lambda 1}^{(i)} z_{i1} + \omega_{\lambda 2}^{(i)} z_{i2} + \ldots . + \omega_i^\mu z_{i\lambda}, \end{aligned}\right\} \quad (i = 1, 2, \ldots . , \mu) \tag{9}$$

where ω_i *and the quantities* $\omega_{kl}^{(i)}$ *are uniform functions of* $(x - a)^{\frac{1}{\mu}}$ *in the vicinity of* $x = a$.

Finally, we must treat the case that $x = a$ is a branch-point, and that $\omega_1, \omega_2, \ldots . , \omega_\lambda$ only differ by factors from ω_1, which are σ^{th} roots of unity (§2). σ was the least common multiple of μ, ν, etc., where these integers gave the number of branches in each cycle of ω's which were permutated by a circuit around $x = a$. Since

$$\omega_1^\sigma = \omega_2^\sigma = \ldots . = \omega_\lambda^\sigma,$$

this reduces the problem to the case of equal roots for a multiple circuit, as solved by equations (9).

§5.—*Analytical Form of the Canonical Functions in the Vicinity of the Branch-point to which they belong.*

In all of the different cases mentioned, it is possible to find analytical expressions for the canonical system of functions which hold in the vicinity of $x = a$.

Let us first consider the case that all of the roots of the characteristic equation are unequal, and uniform in the vicinity of $x = a$. Let $f(x)$ be uniform in the vicinity of $x = a$, and put

$$z = e^{f(x) \log (x-a)}.$$

Then, making a positive circuit around $x = a$,

$$\bar{z} = e^{2\pi i f(x)} z.$$

Putting therefore

$$f(x) = \frac{1}{2\pi i} \log \omega, \quad z = e^{\frac{1}{2\pi i} \log \omega \log (x-a)} \tag{10}$$

we find

$$\bar{z} = \omega z, \tag{10a}$$

if $\log\omega$ is uniform for $x=a$, i. e. if $x-a$ is neither a zero nor an infinity of finite multiplicity of ω. If a is an essential singularity, the expression (10) will not in general verify (10a). It will however do so, as is at once seen, if in the vicinity of $x=a$, ω can be expressed in the form

$$\omega = e^{P(x-a)+P_1\left(\frac{1}{x-a}\right)},$$

for then $\log\omega$ is uniform.

If a is a zero or infinity of ω, expression (10) must be modified. Suppose that it is a zero of multiplicity μ, so that

$$\omega = (x-a)^\mu\omega',$$

where ω' is neither zero nor infinite for $x=a$. If we put

$$z_1 = e^{\frac{1}{2\pi i}\log\omega'\log(x-a)},$$

then, if z is the required function for which $\bar{z}=\omega z$,

$$\bar{z}_1 = \omega' z_1, \quad \left(\frac{\bar{z}}{z_1}\right) = (x-a)^\mu\frac{z}{z_1},$$

or putting $\eta = \dfrac{z}{z_1}$,

$$\eta = (x-a)^\mu\eta.$$

Now let

$$z_2 = e^{\frac{\mu}{4\pi i}[\log(x-a)]^2}.$$

Then

$$\bar{z} = \overline{e^{\frac{\mu}{4\pi i}}}^{[\log(x-a)^2+4\pi i\log(x-a)-4\pi^2]} = z_2(x-a)^\mu e^{-\frac{\mu\pi}{i}}.$$

Therefore if z_3 denotes η/z_2,

$$\bar{z}_3 = e^{-\mu\pi i}z_3,$$

which shows we may put

$$z_3 = (x-a)^{-\frac{\mu}{2}}.$$

Therefore we find

$$z = z_1z_2z_3 = e^{\frac{1}{2\pi i}\log(x-a)[\log\omega'+\frac{\mu}{2}\log(x-a)-\mu\pi i]}. \tag{11}$$

If $x=a$ is an infinity of multiplicity ν, μ in this expression must be replaced by $-\nu$.

Now let $E(\omega)$ denote either (10) or (11), as the case demands. If the roots of the characteristic equation are all unequal,

$$\left[\overline{\frac{z_i}{E(\omega_i)}}\right] = \frac{z_i}{E(\omega_i)},$$

i. e. denoting this quotient by $\phi_i(x)$, $\phi_i(x)$ is uniform in the vicinity of $x = a$, and can therefore be developed in a series of integral powers of $x - a$ containing in general an infinite number of positive and negative exponents. *We will therefore have*

$$z_i = E(\omega_i)\,\phi_i(x). \qquad (i = 1, 2, \ldots, n) \qquad (12)$$

We need not give the detailed proof of the following theorem, as it can be obtained in a manner which is familiar to those interested in linear differential equations.

If ω_1 *is a multiple root of the characteristic equation, uniform in the vicinity of* $x = a$, *occurring* λ *times, the canonical functions belonging to this group of equal roots, whose behavior is expressed by equations* (8), *can be represented in the form*

$$\left.\begin{array}{l} z_1 = E(\omega_1)\,\phi_{11}(x), \\ z_2 = E(\omega_1)[\phi_{21}(x) + \phi_{22}(x)\log(x-a)], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ z_\lambda = E(\omega_1)[\phi_{\lambda 1}(x) + \phi_{\lambda 2}(x)\log(x-a) + \ldots + \phi_{\lambda\lambda}(x)\log(x-a)^{\lambda-1}], \end{array}\right\} \quad (13)$$

where all of the ϕ *functions are uniform for* $x = a$. *If* ω_1 *is absolutely uniform, all of the* ϕ *functions can be expressed in homogeneous linear functions of* $\phi_{11}, \phi_{21}, \ldots, \phi_{\lambda 1}$ *with absolutely uniform coefficients. The coefficients of the highest powers of the logarithm, in particular, differ from* ϕ_{11} *only by an absolutely uniform factor.*

Now let $x = a$ be a branch-point for the μ interchanging branches $\omega_1, \omega_2, \ldots, \omega_\mu$. The canonical functions in this case are those for which

$$\bar{z}_1 = \omega_2 z_2, \; \ldots, \; \bar{z}_{\mu-1} = \omega_\mu z_\mu, \quad \bar{z}_\mu = \omega_1 z_1,$$

if the quotient of no two ω's is a μ^{th} root of unity. The expressions

$$e^{\frac{1}{2\pi i}\log\omega_i \log(x-a)} \qquad (i = 1, 2, \ldots, \mu)$$

have the same property if $\omega_1, \ldots, \omega_\mu$ do not vanish or become infinite for $x = a$, and a slight modification of (11) gives expressions which have the same property if $\omega_1, \ldots, \omega_\mu$ are zero or infinite for $x = a$. It is only necessary to multiply (11) by $(x-a)^{\pm\lambda}$ if λ is the multiplicity of the zero or pole.

Again calling these functions $E(\omega_i)$ in either case, we have

$$\left[\overline{\frac{z_1}{E(\omega_1)}}\right] = \frac{z_2}{E(\omega_2)}, \quad \left[\overline{\frac{z_2}{E(\omega_2)}}\right] = \frac{z_3}{E(\omega_3)}, \; \ldots, \; \left[\overline{\frac{z_\mu}{E(\omega_\mu)}}\right] = \frac{z_1}{E(\omega_1)},$$

i. e. if we put

$$\frac{z_i}{E(\omega_i)} = \psi_i(x),$$

$\psi_1(x), \ldots, \psi_\mu(x)$ are permutated by a circuit around $x = a$. *They are therefore the roots of an equation of degree μ whose coefficients are uniform in the vicinity of $x = a$, and $z_1, \ldots, z_\mu$ are represented by*

$$z_i = E(\omega_i)\,\psi_i(x). \qquad (i = 1, 2, \ldots, \mu) \tag{14}$$

In the vicinity of a, $\psi_i(x)$ will be developed in a series of positive and negative powers of $(x-a)^{\frac{1}{\mu}}$. If

$$\psi_1(x) = P\left[(x-a)^{\frac{1}{\mu}}\right] + P_1\left[\frac{1}{(x-a)}^{\frac{1}{\mu}}\right],$$

then

$$\psi_i(x) = P\left[(x-a)^{\frac{1}{\mu}}\alpha^{i-1}\right] + P_1\left[\frac{1}{(x-a)}^{\frac{1}{\mu}}\alpha^{i-1}\right],$$

where α is a primitive μ^{th} root of unity, namely,

$$\alpha = e^{\frac{2\pi i}{\mu}}.$$

If $x = a$ is a branch-point for the root ω_1, and if this root, and of course also the other roots belonging to the same cycle $\omega_2, \ldots, \omega_\mu$, occurs more than once, say λ times, there will be λ canonical functions corresponding to the root ω_1,

$$\left.\begin{aligned}
z_{11} &= E(\omega_1)\,\psi_{11}(x), \\
z_{12} &= E(\omega_1)\left[\psi_{21}(x) + \psi_{22}(x)\log(x-a)\right], \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
z_{1\lambda} &= E(\omega_1)\left[\psi_{\lambda 1}(x) + \psi_{\lambda 2}(x)\log(x-a) + \cdots + \psi_{\lambda\lambda}(x)\log(x-a)^{\lambda-1}\right],
\end{aligned}\right\} \tag{15}$$

where all of the ψ functions are uniform functions of $t = (x-a)^{\frac{1}{\mu}}$ in the vicinity of $x = a$. If ω_1 is an absolutely uniform function of $(x-a)^{\frac{1}{\mu}}$, all of the ψ functions can be expressed as homogeneous linear functions of $\psi_{11}(x), \ldots, \psi_{\lambda 1}(x)$ with coefficients which are absolutely uniform functions of $(x-a)^{\frac{1}{\mu}}$.

A similar system exists for the $\lambda(\mu-1)$ roots equal to $\omega_2, \omega_3, \ldots, \omega_\mu$ respectively.

If finally $\omega_1, \omega_2, \ldots, \omega_\nu$ are roots of the characteristic equation whose quotients are roots of unity, there exists a system of ν canonical functions of the same form as (15).

§6.—*Cogredient* Λ *functions, and functions of the same kind and class.*

We will call two Λ functions $(y_1, \ldots, y_n)$ and $(y'_1, \ldots, y'_n)$ *cogredient* if they have the same branch-points and the same group of substitutions. This term, borrowed from the theory of invariants, was used in the theory of linear differential equations as I believe for the first time by Klein. It is a generalization of the notions of *class* and *kind* which are due to Riemann and Poincaré.

We will say that two Λ functions belong to the same *kind** if they have not only the same branch-points and substitution group, but also have those singular points in common in whose vicinity they are uniform, but in which they have no definite, finite or infinite value.†

If, finally, for two Λ functions of the same kind the poles also coincide, they are said to belong to the same class.

We will here be principally concerned with the most general of these conceptions, which is that of cogrediency.

If $(y_1, y_2, \ldots, y_n)$, $(y'_1, y'_2, \ldots, y'_n)$, $\ldots$, $(y_1^{(\lambda-1)}, y_2^{(\lambda-1)}, \ldots, y_n^{(\lambda-1)})$ are λ systems of cogredient Λ functions, obviously the system

$$y_i^{(\lambda)} = u_0 y_i + u_1 y'_i + \ldots + u_{\lambda-1} y^{(\lambda-1)}, \qquad (i = 1, 2, \ldots, n) \tag{16}$$

where $u_0, u_1, \ldots, u_{\lambda-1}$ are any uniform functions, is also cogredient with $y_1, \ldots, y_n$.

If between $\lambda + 1$ systems of cogredient Λ functions there is no relation of the form (16), they are said to be *independent. Then the theorem holds that there can be no more than n independent cogredient systems of* Λ *functions.*

For if $(y_1, \ldots, y_n), \ldots, (y_1^{(n)}, \ldots, y_n^{(n)})$ are $n+1$ systems of cogredient Λ functions, and we define $g_1, \ldots, g_n$ by

$$y_i + g_1 y'_i + \ldots + g_n y_i^{(n)} = 0, \qquad (i = 1, 2, \ldots, n) \tag{17}$$

it follows at once that $g_1, \ldots, g_n$ are uniform functions of x. For $g_1, \ldots g_n$ are equal to the quotients of the determinants which are obtained from the matrix

$$\begin{Vmatrix} y_1, & y'_1, & \ldots, & y_1^{(n)} \\ \ldots & \ldots & \ldots & \ldots \\ y_n, & y'_n, & \ldots, & y_n^{(n)} \end{Vmatrix}$$

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* We have translated M. Poincaré's "espèce" by kind.

† Called by Fuchs "Unbestimmtheitsstellen."

by omitting always one vertical line. Now when x describes a circuit around a_i each determinant is multiplied by the determinant of the substitution A_i. Their quotients are therefore uniform.

This result is perfectly analogous to Riemann's theorem, and can be stated as follows:

Any $n+1$ *systems of cogredient* Λ *functions verify a homogeneous linear relation with uniform coefficients.*

The same is true of $n+1$ systems of the same kind or class, except that the uniform coefficients must then be subjected to certain conditions, as in the following theorem:

If the development of the canonical functions in the vicinity of every singular point, and for all of the $n+1$ systems, contains only a finite number of negative powers, the coefficients of the linear relation are rational functions.

§7.—*Linear Differential Equations.*

When the fundamental substitutions have constant coefficients, $\left(\frac{dy_1}{dx}, \frac{dy_2}{dx}, \ldots, \frac{dy_n}{dx}\right)$ are of the same class as $(y_1, \ldots, y_n)$, and this immediately shows that $(y_1, \ldots, y_n)$ are integrals of a linear differential equation with uniform coefficients.

But there are other cases in which Λ functions are connected with linear differential equations. Let S denote any substitution with uniform coefficients, and let us put

$$y_i = S(\eta_i), \quad \eta_i = S^{-1}(y_i).$$

Let $\mathfrak{A}_k$ be the substitution which $\eta_1, \ldots, \eta_n$ suffer if x makes a circuit around $x = a_k$. Then

$$A_k = S\mathfrak{A}_k S^{-1}.$$

Now if all the substitutions $\mathfrak{A}_k$ have constant coefficients, $\eta_1, \ldots, \eta_n$ are a fundamental system of integrals of a linear differential equation, and $y_1, \ldots, y_n$ therefore verify a system of n linear differential equations of the n^{th} order. If therefore all of the substitutions A_k can be expressed in the form

$$A_k = S\mathfrak{A}_k S^{-1},$$

where $\mathfrak{A}_k$ has constant coefficients, and we put

$$\eta_i = S^{-1}(y_i),$$

then $\eta_1, \ldots\ldots, \eta_n$ constitute a fundamental system of integrals for a linear differential equation of the n^{th} order.*

It may also happen that certain linear homogeneous combinations of $(y_1, \ldots, y_n)$ and of the derivatives of $y_1, \ldots, y_n$ up to a certain order, can be found such that they are cogredient with $y_1, \ldots, y_n$. If those combinations are repeated n times, a system of linear differential equations is obtained which $y_1, \ldots, y_n$ must verify.

§8.—*Case that all Substitutions are Transformed into the Canonical Form by the same Transformation.*

The results of §5 obviously prove the existence of Λ functions having only two branch-points by giving their complete expression, the branch-points being $x = a$ and $x = \infty$ in the formulæ. But instead of the latter, any other point may of course be taken.† The functions $z_1, \ldots, z_n$ which are canonical for $x = a$, are also canonical for $x = b$.

In all cases in which the functions $z_1, \ldots, z_n$ are canonical for all singular points $a_1, a_2, \ldots, a_m$ at the same time, we can construct our functions without difficulty. Let us confine ourselves to this special case, which will be useful by showing in a very obvious manner certain relations which in the more general cases remain more or less obscure.

Our case will be even more special than this. Let all characteristic equations have unequal, absolutely uniform roots. In addition to the m branch-points, $a_1, a_2, \ldots, a_m$ we will take $x = \infty$ as branch-point (in general) with its substitution A_∞ so chosen that the relation is verified:

$$A_1 A_2 \ldots A_m A_\infty = 1.$$

The n roots of the characteristic equation belonging to a_i are to be denoted by $\omega_{i1}, \omega_{i2}, \ldots, \omega_{in}$. Then $y_1, \ldots, y_n$ will be homogeneous linear functions of

$$z_k = \prod_{i=1}^{m} E(\omega_{ik})\,\phi_k(x), \qquad (k = 1, 2, \ldots, n)$$

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*I have been informed that Halphen has noticed this, but I have not been able to find the reference in the books at my disposal.

†By a linear transformation.

with uniform coefficients. $\phi_k(x)$ are uniform functions. This shows that these functions as determined by their group still contain n arbitrary uniform functions.

Since each z function is a homogeneous linear function of $y_1, \ldots, y_n$ with uniform coefficients, $z_1, \ldots, z_n$ must be uniform everywhere except in the vicinity of $a_1, a_2, \ldots, a_m, \infty$. Now let b be a point which coincides with neither of these points, and let $x = b$ be a zero or infinity of some of the ω_{ik} functions. For briefness we will omit the subscript k, thus

$$z = \prod_{i=1}^{m} E(\omega_i)\,\phi(x).$$

If ω_i has a_i for a zero or pole (counting the latter as a negative zero), we have

$$E(\omega_i) = e^{\frac{1}{2\pi i}\log(x-a_i)\left[\log\omega_i + \frac{\mu_i}{2}\log(x-a_i) - \mu\pi i\right]},$$

and this holds also if a_i is no zero of ω_i, if then we put $\omega_i' = \omega_i$, $\mu_i = 0$.

Let $x = b$ be a zero of ω_i of multiplicity ν_i. Then

$$\omega_i' = (x-b)^{\nu_i}\omega_i'',$$

where ω_i'' is neither zero nor infinite for $x = b$. But then in general b will become a branch-point for z. z will be uniform in the vicinity of $x = b$ only if the coefficient of $\log(x-b)$ in the exponential vanishes, i. e. if

$$\sum_{i=1}^{m} \nu_i \log(x-a_i) = 0,$$

which is only possible if $\nu_i = 0$, i. e. the points a_i and the multipliers ω_i cannot both be taken arbitrarily. If the singular points a_i are given, the multipliers must be so chosen that they have neither zeros nor poles which are different from these points. But, as is seen in the same way, a_k also can neither be a zero nor a pole of ω_i except for $i = k$.

This suffices to show that the zeros and poles of the roots of the characteristic equations merit especial attention. The results of this paragraph can partly be generalized, but it is not our intention to push this investigation any further at present.

§9.—*Generalized Hypergeometric Functions.*

It can be easily shown that the theorems of Gauss for the hypergeometric series $F(\alpha, \beta, \gamma, x)$, such as the "relationes inter functiones contigues," the development into a continued fraction, the criteria of convergence, the relations between functions having as fourth element x and $1-x$* and the representation by definite integrals, have the character of identities and do not essentially rest upon the assumption that α, β, γ are independent of x. If then α, β, γ are taken as uniform functions of x, which is the generalization which we have in mind, these theorems will remain true. The substitutions which two functions, for instance those defined in the vicinity of $x=0$ by

$$\begin{cases} y_1 = F(\alpha, \beta, \gamma, x), \\ y_2 = x^{1-\gamma} F(\alpha+1-\gamma,\ \beta+1-\gamma,\ 2-\gamma,\ x), \end{cases}$$

undergo when x makes circuits around $x=0$, $x=1$ and $x=\infty$ will then have the same form as when α, β, γ are constants, but their coefficients are now functions of x. Of course y_1 and y_2 are no longer solutions of a linear differential equation in general.

The validity of the relations between the above functions y_1 and y_2 and hypergeometric series, whose fourth element is $1-x$, is especially important from our point of view. The proof usually adopted which depends upon the differential equation is of course unsuitable for the general case. But we can immediately apply that method of proof which starts from the definition by a definite integral.

The relation

$$F(\alpha, \beta, \gamma, x) = \frac{\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}du}{\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}du}, \tag{18}$$

in which the denominator equals $B(\beta, \gamma-\beta)$, remains true if α, β, γ are functions of x, provided that

$$\Re(\beta) > 0, \quad \Re(\gamma-\beta) > 0, \quad \Re(\alpha) < 1, \tag{19}$$

i. e. for such values of x for which these inequalities hold. But if these inequalities are not fulfilled we will have to consider the double-loop integrals which have a definite sense, whatever values α, β, γ may have.

We could essentially follow the course of reasoning given by Klein in his "Vorlesung, Ueber die hypergeometrische Funktion," Göttingen, 1894. Only some obvious points need closer consideration if we wish to determine the function theoretic character of our functions in that way directly.

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* Called by Fuchs "Uebergangssubstitutionen."

It suffices to give the result. The double-loop integrals

$$\left.\begin{aligned} y_1 &= \int_{(0,\,x)} u^{\alpha-\gamma}(u-1)^{\gamma-\beta-1}(x-u)^{-\alpha}du\,, \\ y_2 &= \int_{(1,\,x)} u^{\alpha-\gamma}(u-1)^{\gamma-\beta-1}(x-u)^{-\alpha}du\,, \end{aligned}\right\} \tag{20}$$

the integration being on a double loop around 0 *and* x, *and around* 1 *and* x *respectively, form the elements of a* Λ *function with the branch-points* 0, 1, ∞. *The substitutions belonging to* 0 *and* 1 *are*

$$A_0 = \begin{pmatrix} e^{-2\pi i\gamma}, & 0 \\ -(1-e^{2\pi i(\gamma-\beta)}), & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1; & -e^{-2\pi i\alpha}(1-e^{2\pi i(\alpha-\gamma)}) \\ 0, & e^{2\pi i(\gamma-\beta-\alpha)} \end{pmatrix}, \tag{21}$$

while, of course,

$$A_\infty = A_0^{-1}A_1^{-1}.^*$$

The points $x = 0, 1, \infty$ are not the only singular points of the Λ function however. But these other singularities are only infinities in whose vicinity the function is uniform. They can be easily found from the representation in terms of the hypergeometric series.

The Λ function (20) will not in general verify an algebraical differential equation with rational coefficients. For if we put $\gamma-\beta-1=0$, y_1 will essentially become equal to an Eulerian integral of the first kind $B(\alpha-\gamma+1, -\alpha+1)$. Putting further $\gamma=1$, and therefore $\beta=0$,

$$y = B(\alpha-\gamma+1, -\alpha+1) = B(\alpha, -\alpha+1) = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)} = \frac{\pi}{\sin\pi\alpha}.$$

This verifies the equation

$$\left(\frac{dy}{d\alpha}\right)^2 = y^4\left(1-\frac{y^2}{\pi^2}\right),$$

whence

$$\left(\frac{dy}{dx}\right)^2 = y^4\left(1-\frac{y^2}{\pi^2}\right)\left(\frac{d\alpha}{dx}\right)^2.$$

Therefore $y, \dfrac{dy}{dx}, \ldots, \dfrac{d^ny}{dx^n}$ can be expressed as algebraical functions of $\sin\pi\alpha, \dfrac{d\alpha}{dx}, \ldots, \dfrac{d^n\alpha}{dx^n}$. Now if there were an algebraic relation between

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[*] For the proof when α, β, γ are constants, see Schleisinger, Hand. d. Lin. Diffgl. vol. II, pp. 460, 468.

$y, \frac{dy}{dx}, \ldots\ldots, \frac{d^n y}{dx^n}$ with coefficients algebraic in x, there would be a similar relation between $\sin \pi\alpha, \frac{d\alpha}{dx}, \ldots\ldots, \frac{d^n\alpha}{dx^n}$. Computing $\sin \pi\alpha$ from this relation, differentiating with respect to x, and eliminating $\sin \pi\alpha$, an algebraical relation between $\frac{d\alpha}{dx}, \ldots\ldots, \frac{d^{n+1}\alpha}{dx^{n+1}}$ results. But we can choose α as uniform function of x, so that this is impossible. For instance, if we put

$$\alpha = \Gamma(x),$$

a function which, according to Mr. Hölder,* verifies no algebraical differential equation with rational coefficients, y will have the same property. In general, then, if α, β, γ are any uniform functions, the Λ function will not verify a differential equation with rational coefficients. It is an entirely different question, however, whether it may verify such an equation with uniform transcendental functions.

We will prove one general theorem about such differential equations in the next paragraph.

The Λ functions found in this paragraph can be easily generalized. The essential property, which must be retained in this generalization, is the representation by definite integrals.

§10.—*Differential Equations.*

Suppose that $(y_1, \ldots\ldots, y_n)$ is a Λ function, and that at least one of the characteristic equations belonging to some one branch-point, has absolutely uniform and distinct roots. Then there exist n functions

$$z_i = \lambda_{i1} y_1 + \lambda_{i2} y_2 + \ldots\ldots + \lambda_{in} y_n, \qquad (22)$$
$$(i = 1, 2, \ldots\ldots, n)$$

where λ_{ik} are uniform functions which, for a circuit around that branch-point a, undergo the substitution

$$\bar{z}_i = \omega_i z_i. \qquad (i = 1, 2, \ldots\ldots, n) \qquad (23)$$

Now suppose that $(y_1, \ldots\ldots, y_n)$ verify a system of algebraical differential equations with uniform coefficients,

$$D_k(y_1, y_2, \ldots\ldots, y_n) = 0. \qquad (k = 1, 2, \ldots\ldots, n) \qquad (24)$$

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* Hölder, Math. Ann., vol. 28, or Moore, Math. Ann., Bd. 48.

Then substituting the values of y_i found from (22), $z_1, \ldots, z_n$ will also verify such a system with uniform coefficients

$$\Delta_k(z_1, z_2, \ldots, z_n) = 0. \quad (k = 1, 2, \ldots, n) \quad (25)$$

But from (25) a system of n differential equations can be deduced, each of which contains only one unknown function z_k. These equations will appear in the form

$$\frac{d^{n_k}z_k}{dx^{n_k}} = F_k\left(x, z_k, \frac{dz_k}{dx}, \ldots, \frac{d^{n_k-1}z_k}{dx^{n_k-1}}\right),$$

where F_k is an algebraic function of $z_k, \ldots, \frac{dx^{n_k-1}z_k}{dx^{n_k-1}}$ and of the coefficients of (25). These equations can therefore be reduced to the form

$$E_k(z_k) = 0, \qquad (k = 1, 2, \ldots, n) \qquad (26)$$

where E_k is a rational integral function of $z_k, \ldots, \frac{d^{n_k}z_k}{dx^{n_k}}$ with coefficients which are uniform functions of x.

Let

$$E(z) = 0 \qquad (27)$$

be any one of these equations. If x describes a circuit around $x = a$, z changes into ωz, where ω is a uniform function, while the coefficients of E remain unaltered. We must therefore also have

$$E(\omega z) = 0, \qquad (28)$$

i. e. ωz must also be a solution of (27).

Let the equation be of the n^{th} order and of the m^{th} degree. We can write for the left member of (27),

$$E(y) = E_m(y) + E_{m-1}(y) + \ldots + E_1(y) + E_0(y), \qquad (29)$$

denoting by E_k a homogeneous polynomial of degree k,

$$E_k(y) = \Sigma p_{i_1 i_2 \ldots i_k} y^{(i_1)} y^{(i_2)} \ldots y^{(i_k)}, \qquad (30)$$
$$(i_1, i_2, \ldots, i_k = 0, 1, 2, \ldots, n)$$

where $p_{i_1 i_2 \ldots i_k}$ are uniform functions.*

In such an assemblage of terms we will call those terms the *highest* for which the sum of the indices

$$i_1 + i_2 + \ldots + i_k$$

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* $y^{(i_\lambda}$ denotes $\frac{d^{(i_\lambda)}y}{dx^{i_\lambda}}$ as usual.

has the greatest value. We always have

$$i_1 + i_2 + \dots + i_k \leqq kn. \qquad (k = 1, 2, \dots, m)$$

Now (29) vanishes for $y = z$ or for $y = \omega z$. Putting $y = \omega z$, we have

$$y^{(i)} = \omega z^{(i)} + \binom{i}{1} \omega' z^{(i-1)} + \dots,$$

which shows that the highest terms in $E_k(\omega z)$ differ from those in $E_k(z)$ only by the factor ω^k. If we divide $E(\omega z)$ by ω^m, which is of course different from zero, the highest terms of $E(z)$ and $\frac{1}{\omega^m} E(\omega z)$ will be identical. If all of the other terms of these two expressions are not identical we will have another differential equation for z,

$$E(z) - \frac{1}{\omega^m} E(\omega z) = 0,$$

in which the highest terms of the m^{th} degree are at least by one lower than in (27). We will say that this second equation has a lower *type* than the first. If, then, z does not verify a differential equation of lower type than (27), a supposition which we can make without loss of generality, the corresponding coefficients of $E(z)$ and $\frac{1}{\omega^m} E(\omega z)$ must be equal.

Now suppose that $p_{k_1 k_2 \dots k_{m-1}} z^{(k_1)} z^{(k_2)} \dots z^{(k_{m-1})}$ is the highest term in $E_{m-1}(z)$. The corresponding term in $E_{m-1}(\omega z) \frac{1}{\omega^m}$ is $p_{k_1 \dots k_{m-1}} z^{(k_1)} z^{(k_2)} \dots z^{(k_{m-1})} \omega^{m-1} \frac{1}{\omega^m}$.

Therefore

$$p_{k_1 k_2 \dots k_{m-1}} = \frac{1}{\omega} p_{k_1 k_2 \dots k_{m-1}},$$

which would give $\omega = 1$ if $p_{k_1 k_2 \dots k_{m-1}}$ did not vanish. But that would make z uniform in the vicinity $x = a$, which is contrary to our supposition. Therefore

$$p_{k_1 k_2 \dots k_{m-1}} = 0.$$

And since this is the highest term of $E_{m-1}(z)$,

$$E_{m-1}(z) = E_{m-2}(z) = \dots = E_0(z) = 0,$$

i. e. the differential equation reduces to

$$E_m(z) = 0.$$

It is homogeneous. But according to (22) it is at once seen that *if the characteristic equation belonging to at least one branch-point has absolutely uniform and distinct roots, the system of differential equations with uniform coefficients, if any exists, which the elements of this* Λ *function verify, consists of homogeneous equations.*

Let us give a simple example. In the generalized hypergeometric series,

$$y = 1 + \frac{\alpha . \beta}{1 . \gamma} x + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{1 . 2 . \gamma(\gamma + 1)} x^2 + \ldots ,$$

let $\beta = \gamma$, and let α be any uniform function. Then

$$y = 1 + \frac{\alpha}{1!} x + \frac{\alpha(\alpha + 1)}{2!} x^2 + \ldots = (1 - x)^{-\alpha} = e^{-\alpha \log [1 - x]} .$$

Therefore

$$\frac{1}{y} \frac{dy}{dx} = + \frac{\alpha}{1 - x} - \frac{d\alpha}{dx} \log(1 - x),$$

$$\frac{1}{y} \frac{d^2y}{dx^2} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 = + \frac{\alpha}{(1 - x)^2} + 2 \frac{d\alpha}{dx} \frac{1}{1 - x} - \frac{d^2\alpha}{dx^2} \log(1 - x).$$

Whence

$$\frac{d^2\alpha}{dx^2} \frac{1}{y} \frac{dy}{dx} - \frac{d\alpha}{dx}\left[\frac{1}{y} \frac{d^2y}{dx^2} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2\right] = \left[\alpha \frac{d^2\alpha}{dx^2} - 2\left(\frac{d\alpha}{dx}\right)^2\right] \frac{1}{1 - x} - \alpha \frac{d^2\alpha}{dx^2} \frac{1}{(1 - x)^2},$$

or

$$y \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 - \frac{\frac{d^2\alpha}{dx^2}}{\frac{d\alpha}{dx}} y \frac{dy}{dx} = y^2 \left[\alpha \frac{\frac{d^2\alpha}{dx^2}}{\frac{d\alpha}{dx}} \frac{1}{(1 - x)^2} - \left(\alpha \frac{\frac{d^2\alpha}{dx^2}}{\frac{d\alpha}{dx}} - 2 \frac{d\alpha}{dx}\right) \frac{1}{1 - x}\right],$$

a homogeneous quadratic equation for y. If α is a rational function, the coefficients are also rational. The equation is a non-homogeneous linear equation for $\log y$. The most general homogeneous quadratic equation of the second order which can be satisfied by Λ functions is such an equation, linear in $\log y$.

§11.—*Conclusion.*

Confining ourselves to the behavior of Λ functions in the vicinity of any one branch-point, we found no difficulty in deducing general results. But we could only study the functions for all values of x by limiting ourselves to such special cases in which we could give analytical expressions defining them. In the case of constant coefficients of the substitutions not very much more has been done, the group is supposed to be given. A direct existence-theorem as required by Riemann's principles has only been given by Klein for a linear differential equation of the second order, the roots of whose determinating fundamental equation are real. We have not been able to find any proof of this kind, but have been satisfied in verifying the existence of such functions expressible as definite integrals.

There are some lines of investigation which seem to be promising. First, we can investigate the Λ functions expressible by definite integrals. Secondly, starting from the conception of cogrediency, we can attempt to formally find systems of combinations of $(y_1, \ldots, y_n)$ and their derivatives, which are cogredient to $(y_1, \ldots, y_n)$ and thus find differential equations which the functions belonging to a certain group must verify, if they exist. Finally, these equations will also be found by continuing the formal process used in §10. The conditions that the differential equations thus found are integrated by Λ functions can then be investigated, as well as the functions themselves. The inversion problems associated with such functions will be of especial interest and importance.

CHICAGO, *Feb.* 27, 1898.

The Partial Differential Equations for the Hyperelliptic Θ- and σ-Functions.

BY OSKAR BOLZA.

The partial differential equation satisfied by the hyperelliptic Θ- and σ-functions, which furnish the recursion formulæ for the expansion of these functions into power series, have been first established by Wiltheiss in a series of papers published in Crelle's Journal* and the Mathematische Annalen.† Two steps can be distinguished in his deductions:

First he establishes a system of partial differential equations for a canonical system of integrals of the first and second kind or their periods, the differentiation taking place either with respect to the roots or with respect to the coefficients of the polynomial on whose square root the integrals depend.

From these preliminary differential equations he derives, in a second step, the final equations for the Θ-functions by several different methods, all of them indirect and rather complex.

The principal object of the present paper is to replace this part of Wiltheiss' work by simpler and more direct proofs. Two such proofs are given in §4 and §5: the first proceeds directly from Weierstrass' definition of the most general Θ-function by means of an exponential series without using any further properties of the Θ-functions; the second starts from the expression of the Θ-functions in terms of the ϑ-functions, and makes use of the well-known partial differential equations of the ϑ-function. In both proofs the work is vastly simplified by the use of the notations and methods of the *theory of matrices*[*] which has already been so successfully applied to the treatment of Θ-functions by Baker in his book on Abelian functions.

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* Bd. 99, p. 236 (1884).

† Bd. 29, p. 272 (1886); Bd. 31, p. 134 (1887); Bd. 33, p. 267 (1888).

The preceding §§1 to 3 contain a simplified proof for the partial differential equations satisfied by the periods of the integrals of the first and second kind. For independent variables I have chosen the branch-points, as Wiltheiss does in his first paper; it seems indeed, from the simple recursion formula given by Brioschi[†] for the expansion of the even σ-functions of two variables, that differentiation with respect to the branch-points will, in the end, furnish simpler results than the Aronhold process which Wiltheiss uses in his later publications.

The two concluding §§6 and 7 contain applications of the previous results to Wiltheiss' *Th*-functions, to that class of σ-functions for which—in Klein's notation—$\mu = 0$, and to the expression of $\Theta(0, 0 \ldots . 0)$ in terms of the branch-points.

§1.—*Notations.*

I use throughout the notations of Weierstrass' Lectures on Hyperelliptic Functions of Winter 1881–82, with the same slight modifications adopted in my former papers "On Weierstrass' Systems of Hyperelliptic Integrals of the First and Second Kind,"[‡] and "On the First and Second Logarithmic Derivatives of Hyperelliptic σ-functions."[§]

Accordingly the *hyperelliptic curve* is denoted by

$$y^2 = R(x) = \sum_{\lambda=0}^{2\rho+2} \binom{2\rho+2}{\lambda} A_\lambda x^{2\rho+2-\lambda} = A_0 \prod_{i=0}^{2\rho+1} (x - a_i), \qquad (1)$$

$$w_\alpha = \int \frac{g_\alpha(x)\, dx}{y} \qquad \alpha = 1, 2, \ldots, \rho$$

are a system of ρ linearly independent *integrals of the first kind.*

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[*] Cayley, "A Memoir on the Theory of Matrices," Phil. Trans., vol. 148. See also Taber, "On the Theory of Matrices," American Journal, vol. XII; Weyr, "Zur Theorie der bilinearen Formen, Monatshefte für Mathematik u. Physik, Bd. 1, and Baker, "Abel's Theorem and the Allied Theory," Art. 189–191, 405–410.

[†] Göttinger Nachrichten, 1890, p. 236.

[‡] Papers read at the International Mathematical Congress, 1893, p. 1.

[§] American Journal of Mathematics, vol. XVII, p. 11.

$F(x, \xi)$ is an integral function of x and ξ of degree $\rho+1$ in either variable, satisfying the three conditions:

$$\left.\begin{aligned} F(\xi, x) &= F(x, \xi), \\ F(\xi, \xi) &= R(\xi) \quad, \\ \left(\frac{\partial F(x, \xi)}{\partial x}\right)_{x=\xi} &= \tfrac{1}{2} R'(\xi) \quad, \end{aligned}\right\} \tag{2}$$

$R'(x)$ denoting the derivative of $R(x)$.

The integral

$$\int \frac{F(x, \xi)\, dx}{(x-\xi)^2 y\eta}$$

—where $\eta^2 = R(\xi)$—is of the second kind; its only poles are the two conjugate points $(\xi, \pm\eta)$, and in the vicinity of these poles an expansion of the following form holds:

$$\int \frac{F(x, \xi)\, dx}{(x-\xi)^2 y\eta} = \frac{\mp 1}{x-\xi} + \mathfrak{P}(x-\xi). \tag{3}$$

If a be a branch-point, the integral

$$\int \frac{F(x, a)\, dx}{(a-a)^2 y} \text{ is again of the second kind,}$$

has but one pole a, and admits in its vicinity the expansion

$$\int \frac{F(x, a)\, dx}{(x-a)^2 y} = -\frac{\sqrt{R'(a)}}{(x-a)^{\frac{1}{2}}} + \mathfrak{P}((x-a)^{\frac{1}{2}}). \tag{4}$$

The polynomials of degree $\rho - 1$ $g_\alpha(x)$, together with the function $F(x, \xi)$, determine uniquely ρ polynomials $g_{\rho+\alpha}(x)$ of degree 2ρ at most by means of the fundamental relation

$$\sum_\alpha \frac{g_{\rho+\alpha}(\xi)\, g_\alpha(x)}{\eta} = \frac{d}{d\xi}\left(\frac{1}{2}\,\frac{\eta}{(x-\xi)}\right) - \frac{F(x, \xi)}{2(x-\xi)^2 \eta}. \tag{5}$$

The ρ integrals

$$w_{\rho+\alpha} = \int \frac{g_{\rho+\alpha}(x)\, dx}{y} \qquad \alpha = 1, 2, \ldots, \rho$$

are of the second kind and constitute, together with the ρ integrals w_α, a *canonical system of associated integrals of the first and second kind.*

In the following developments no special assumption concerning this canonical system is made except that we suppose *the polynomials* $g_\alpha(x)$ *to be independent of the branch-points* a_λ.

The letters $\alpha, \beta, \gamma \dots$ are always used for summation indices running from 1 to ρ; $\sum\limits_{\alpha,\beta}$ means $\sum\limits_{\alpha=1}^{\rho}\sum\limits_{\beta=1}^{\rho}$, whereas the symbol $\sum\limits_{(\alpha,\beta)}$ denotes summation over all the $\frac{\rho(\rho-1)}{2}$ two-combinations of the numbers $1, 2, \dots, \rho$.

§2.—*The Derivatives of the Integrals of the First Kind with Respect to a Branch-point.**

Let a be any one of the $2\rho+2$ branch-points a_λ; we propose in this § to determine the partial derivative of the integral w_α with respect to a.

Following the example of Klein and Wiltheiss, we unite the ρ integrals w_α into one by introducing a parameter ξ as follows: Let $h_1(\xi), h_2(\xi), \dots, h_\rho(\xi)$ be ρ polynomials in ξ defined by the equation

$$(x-\xi)^{\rho-1} = \sum_\beta g_\beta(x)\, h_\beta(\xi). \tag{6}$$

The $h_\beta(\xi)$'s will be of degree $\rho-1$ at most, linearly independent, and independent of a. Hence, if the path of integration be likewise independent of a,

$$\sum_\beta h_\beta(\xi)\frac{\partial w_\beta}{\partial a} = \int \frac{\partial}{\partial a}\frac{(x-\xi)^{\rho-1}}{y}\,dx = \tfrac{1}{2}\int\frac{(x-\xi)^{\rho-1}\,dx}{(x-a)\,y}.$$

This is an integral of the second kind; in order to express it in terms of the integrals w_α, $w_{\rho+\alpha}$, notice that its only pole is a and the expansion in the vicinity of a:

$$-\frac{(a-\xi)^{\rho-1}(x-a)^{-\frac{1}{2}}}{\sqrt{R'(a)}} + \dots.$$

Hence, according to (4), the integral

$$\int\left[\frac{1}{2}\frac{(x-\xi)^{\rho-1}}{x-a} - \frac{(a-\xi)^{\rho-1}}{R'(a)}\frac{F(x,a)}{(x-a)^2}\right]\frac{dx}{y}$$

is of the first kind; the expression in the bracket [ ] is therefore a polynomial in x of degree $\rho - 1$ at most, and the same is true with respect to ξ. We may therefore write

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* Compare for this § also Schroeder, "Ueber den Zusammenhang hyperelliptischer σ- und ϑ-Functionen," Diss. Göttingen, 1890, §9.

$$\tfrac{1}{2}\frac{(x-\xi)^{\rho-1}}{x-a} - \frac{(a-\xi)^{\rho-1}}{R'(a)}\,\frac{F(x,\,a)}{(x-a)^2} = \sum_{\alpha,\,\beta} \varkappa_{\alpha\beta}\, g_\alpha(x)\, h_\beta(\xi), \tag{7}$$

the $\varkappa_{\alpha\beta}$'s being independent of x and ξ.

The integral of the second kind,

$$\int \frac{F(x,\,a)\,dx}{(x-a)^2\,y},$$

is at once expressible in terms of the integrals $w_{\rho+\alpha}$ by means of (5), viz.

$$\int \frac{F(x,\,a)\,dx}{(x-a)^2\,y} = \frac{y}{a-x} - 2\sum_\alpha g_\alpha(a)\, w_{\rho+\alpha}.$$

We thus obtain

$$\frac{\partial}{\partial a}\int \frac{(x-\xi)^{\rho-1}\,dx}{y} = \int \left[\tfrac{1}{2}\frac{(x-\xi)^{\rho-1}}{x-a} - \frac{(a-\xi)^{\rho-1}}{R'(a)}\,\frac{F(x,\,a)}{(x-a)^2}\right]\frac{dx}{y}$$
$$- \frac{(a-\xi)^{\rho-1}}{R'(a)}\left[\frac{y}{a-x} - 2\sum_\alpha g_\alpha(a)\, w_{\rho+\alpha}\right]. \tag{8}$$

To obtain from (8) the derivatives $\dfrac{\partial w_\alpha}{\partial a}$, it only remains to arrange, by means of (6) and (7), both sides according to the functions $h_\beta(\xi)$ and equate corresponding coefficients.

If, in particular, we suppose the path of integration to be closed in the Riemann surface, and denote by

$$2\omega_\alpha \text{ and } 2\eta_\alpha,$$

the corresponding periods of the integrals w_α and $w_{\rho+\alpha}$ respectively, we obtain the

Theorem I:

The simultaneous periods $2\omega_\alpha$, $2\eta_\alpha$ of the integrals w_α, $w_{\rho+\alpha}$ satisfy the differential equations

$$\frac{\partial \omega_\alpha}{\partial a} = \sum_\beta \varkappa_{\beta\alpha}\,\omega_\beta - \sum_\beta \frac{2 g_\alpha(a)\, g_\beta(a)}{R'(a)}\,\eta_\beta, \tag{A}$$

the quantities $\varkappa_{\alpha\beta}$ being defined by (6) *and* (7).

§3.—*The Derivatives of the Integrals of the Second Kind with respect to a Branch-point.*

To obtain the derivatives $\frac{\partial w_{\rho+\alpha}}{\partial a}$ we start from equation (5), in which we interchange x and ξ. Differentiating with respect to a and integrating with respect to x along a path independent of a, we obtain

$$\sum_{\alpha} g_{\alpha}(\xi)\frac{\partial w_{\rho+\alpha}}{\partial a} = \tfrac{1}{2}\frac{\partial}{\partial a}\,\frac{y}{\xi - x} + \int\left[-\tfrac{1}{4}\,\frac{F(x,\,\xi)}{(x-a)(x-\xi)^2} - \tfrac{1}{2}\,\frac{\frac{\partial F(x,\,\xi)}{\partial a}}{(x-\xi)^2}\right]\frac{dx}{y}. \quad (9)$$

The integral on the right-hand side is of the second kind; its poles are $x = a$ and the two conjugate points $(\xi,\,\pm\eta)$. In the vicinity of a the expansion begins with the term

$$\tfrac{1}{2}\,\frac{F(a,\,\xi)}{(\xi-a)^2\sqrt{R'(a)}}\,(x-a)^{-\frac{1}{2}},$$

which is at the same time the first term in the expansion of

$$-\tfrac{1}{2}\,\frac{F(a,\,\xi)}{(\xi-a)^2\,R'(a)}\int\frac{F(x,\,a)\,dx}{(x-a)^2\,y},$$

according to (4).

In the vicinity of $(\xi,\,\pm\eta)$ the first term of the expansion is

$$\tfrac{1}{4}\,\frac{\mp\eta}{\xi-a}\cdot\frac{1}{x-\xi},$$

since $$F(x,\,\xi) = R(\xi) + \tfrac{1}{2}R'(\xi)(x-\xi) + \ldots,$$
and therefore

$$\frac{\partial F(x,\,\xi)}{\partial a} = \frac{\partial R(\xi)}{\partial a} + \ldots = -\frac{R(\xi)}{\xi-a} + \ldots$$

But this is, according to (3), at the same time the first term in the expansion of

$$\frac{1}{4(\xi-a)}\int\frac{F(x,\,\xi)\,dx}{(x-\xi)^2\,y}.$$

Hence, if we put for shortness,

$$\Lambda(x,\,\xi) = \tfrac{1}{4}\left(\frac{1}{x-a} + \frac{1}{\xi-a}\right)\frac{F(x,\,\xi)}{(x-\xi)^2} + \tfrac{1}{2}\,\frac{\frac{\partial F(x,\,\xi)}{\partial a}}{(x-\xi)^2} + -\tfrac{1}{2}\,\frac{F(x,\,a)\,F(\xi,\,a)}{R'(a)(x-a)^2(\xi-a)^2}, \quad (10)$$

the integral

$$\int \frac{\Lambda(x, \xi)\, dx}{y}$$

is of the first kind and consequently $\Lambda(x, \xi)$ a polynomial in x of degree $\rho - 1$ at most, and since

$$\Lambda(x, \xi) = \Lambda(\xi, x), \tag{11}$$

the same is true with respect to the variable ξ. We may therefore write

$$\Lambda(x, \xi) = \sum_{\alpha, \beta} \lambda_{\alpha\beta}\, g_\alpha(x)\, g_\beta(\xi), \tag{12}$$

where $\lambda_{\alpha\beta}$ is independent of x and ξ, and moreover

$$\lambda_{\alpha\beta} = \lambda_{\beta\alpha}. \tag{13}$$

Equation (9) may now be written

$$\sum_\alpha g_\alpha(\xi) \frac{\partial w_{\rho+\alpha}}{\partial a} = \frac{\partial}{\partial a}\left(\tfrac{1}{2} \frac{y}{\xi - x}\right) - \int \frac{\Lambda(x, \xi)\, dx}{y}$$
$$+ \tfrac{1}{4} \int \frac{F(x, \xi)\, dx}{(\xi - a)(x - \xi)^2 y} - \tfrac{1}{2} \int \frac{F(x, a)\, F(\xi, a)\, dx}{R'(a)(x - a)^2 (\xi - a)^2 y},$$

and if we express the two last integrals in terms of the integrals $w_{\rho+\alpha}$ by means of (5) and let the path of integration be the same closed path as in §2, we obtain

$$\sum_\alpha g_\alpha(\xi) \frac{\partial \eta_\alpha}{\partial a} = -\sum_{\alpha, \beta} \lambda_{\alpha\beta}\, g_\beta(\xi)\, w_\alpha - \sum_\beta \eta_\beta \left[\frac{g_\beta(\xi)}{2(\xi - a)} - \frac{F(\xi, a)\, g_\beta(a)}{R'(a)\,(\xi - a)^2}\right].$$

From (2) it follows that the expression

$$\chi_\beta(\xi) = \frac{g_\beta(\xi)}{2(\xi - a)} - \frac{F(\xi, a)\, g_\beta(a)}{R'(a)(\xi - a)^2}$$

remains finite in $\xi = a$, and is therefore an integral function of ξ of degree $\rho - 1$. In order to arrange it according to the functions $g_\alpha(\xi)$, multiply by $h_\beta(t)$ and sum from $\beta = 1$ to $\beta = \rho$:

$$\sum_\beta \chi_\beta(\xi)\, h_\beta(t) = \frac{(\xi - t)^{\rho-1}}{2(\xi - a)} - \frac{(a - t)^{\rho-1} F(\xi, a)}{R'(a)(\xi - a)^2} = \sum_{\alpha, \beta} \varkappa_{\alpha\beta}\, g_\alpha(\xi)\, h_\beta(t),$$

according to (6) and (7). Hence follows

$$\chi_\beta(\xi) = \sum_\alpha \varkappa_{\alpha\beta}\, g_\alpha(\xi). \tag{14}$$

Thus we obtain the

Theorem II.

The simultaneous periods $2\omega_\alpha$, $2\eta_\alpha$ *of the integrals* w_α, $w_{\rho+\alpha}$ *satisfy also the differential equations*

$$\frac{\partial \eta_\alpha}{\partial a} = -\sum_\beta \lambda_{\alpha\beta}\,\omega_\beta - \sum_\beta \varkappa_{\alpha\beta}\,\eta_\beta, \tag{B}$$

the quantities $\varkappa_{\alpha\beta}$ *and* $\lambda_{\alpha\beta}$ *being defined by* (7) *and* (12).

Let now

$$\left.\begin{matrix} 2\omega_{\alpha\beta}, & 2\omega'_{\alpha\beta} \\ 2\eta_{\alpha\beta}, & 2\eta'_{\alpha\beta} \end{matrix}\right\} \tag{15}$$

be a canonical system of periods of the integrals w_α and $w_{\rho+\alpha}$ respectively. The differential equations satisfied by these periods, according to (A) and (B), may be written in an abbreviated form by the use of the notations of the theory of matrices. Write in a general way

$$M = \{m_{\alpha\beta}\},$$

the first index always referring to the row, the second to the column; let further $\bar{M}$ denote, as usual, the matrix derived from M by interchanging the rows with the columns ("transverse of M"). And write finally

$$\left\{\frac{\partial \omega_{\alpha\beta}}{\partial a}\right\} = \delta\Omega, \quad \left\{\frac{\partial \omega'_{\alpha\beta}}{\partial a}\right\} = \delta\Omega',$$

$$\left\{\frac{\partial \eta_{\alpha\beta}}{\partial a}\right\} = \delta\mathrm{H}, \quad \left\{\frac{\partial \eta'_{\alpha\beta}}{\partial a}\right\} = \delta\mathrm{H}',$$

$$\Gamma = \left\{\frac{2g_\alpha(a)\,g_\beta(a)}{R'(a)}\right\}.$$

With these notations and with Cayley's agreements concerning multiplication and addition of matrices, the $4\rho^2$ differential equations obtained by applying (A) and (B) successively to the 2ρ period-paths, may be combined into the following four matrix equations:

$$\left.\begin{aligned} \delta\Omega &= \bar{\mathrm{K}}\Omega - \Gamma\mathrm{H}, \\ \delta\Omega' &= \bar{\mathrm{K}}\Omega' - \Gamma\mathrm{H}', \end{aligned}\right\} \tag{A'}$$

$$\left.\begin{aligned} \delta\mathrm{H} &= -\Lambda\Omega - \mathrm{K}\mathrm{H}, \\ \delta\mathrm{H}' &= -\Lambda\Omega' - \mathrm{K}\mathrm{H}'. \end{aligned}\right\} \tag{B'}$$

§4.—*The Partial Differential Equation for the Θ-Functions: First Proof.*

Weierstrass' function*

$$\Theta\,(u_1, u_2, \ldots\ldots, u_\rho;\ m, n)$$

associated with the canonical system of integrals w_α, $w_{\rho+\alpha}$ and the canonical system of periods (15) is defined as follows:

Let the three matrices A, B, T be defined by the equations

$$2\mathrm{A}\Omega = \mathrm{H}, \quad 2\mathrm{B}\Omega = 1, \quad 2\mathrm{B}\Omega' = \mathrm{T}. \tag{16}$$

Since†

$$\overline{\Omega}\mathrm{H} = \overline{\mathrm{H}}\Omega, \quad \Omega\overline{\Omega}' = \Omega'\overline{\Omega},$$

$$\left.\begin{array}{ll} \text{it follows that} & \overline{\mathrm{A}} = \mathrm{A}, \quad \overline{\mathrm{T}} = \mathrm{T}, \\ \text{that is} & a_{\beta\alpha} = a_{\alpha\beta}, \quad \tau_{\beta\alpha} = \tau_{\alpha\beta}. \end{array}\right\} \tag{27}$$

Then

$$\Theta\,(u_1, u_2, \ldots\ldots, u_\rho;\ m, n)$$

$$= \sum_{\nu_1, \nu_2, \ldots\ldots, \nu_\rho} e^{\phi\,(u_1, u_2, \ldots\ldots, u_\rho;\ \nu_1 + n_1, \nu_2 + n_2, \ldots\ldots, \nu_\rho + n_\rho) + 2\pi i \sum\limits_{\alpha} m_\alpha\,(\nu_\alpha + n_\alpha)}, \tag{18}$$

where

$$\phi\,(u_1 \ldots\ldots u_\rho, \nu_1 \ldots\ldots \nu_\rho) = \sum_{\alpha,\,\beta} a_{\alpha\beta} u_\alpha u_\beta + 2\pi i \sum_{\alpha,\,\beta} b_{\alpha\beta} \nu_\alpha u_\beta + \pi i \sum_{\alpha,\,\beta} \tau_{\alpha\beta} \nu_\alpha \nu_\beta,$$

and the indices $\nu_1, \nu_2, \ldots\ldots, \nu_\rho$ take independently all integer values from $-\infty$ to $+\infty$.

The object of the present § is to express the derivative of Θ with respect to a branch-point a in terms of Θ and its first and second derivatives with respect to the u_α's. Since a is implicitly contained in the quantities $a_{\alpha\beta}$, $b_{\alpha\beta}$, $\tau_{\alpha\beta}$, we have‡

$$\frac{\partial\Theta}{\partial a} = \sum_{\alpha} \frac{\partial\Theta}{\partial a_{\alpha\alpha}} \frac{\partial a_{\alpha\alpha}}{\partial a} + \sum_{(\alpha,\,\beta)} \frac{\partial\Theta}{\partial a_{\alpha\beta}} \frac{\partial a_{\alpha\beta}}{\partial a} + \sum_{(\alpha,\,\beta)} \frac{\partial\Theta}{\partial b_{\alpha\beta}} \frac{\partial b_{\alpha\beta}}{\partial a}$$

$$+ \sum_{\alpha} \frac{\partial\Theta}{\partial \tau_{\alpha\alpha}} \frac{\partial \tau_{\alpha\alpha}}{\partial a} + \sum_{(\alpha,\,\beta)} \frac{\partial\Theta}{\partial \tau_{\alpha\beta}} \frac{\partial \tau_{\alpha\beta}}{\partial a}. \tag{19}$$

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* "Lectures on Hyperelliptic Functions," and Schottky, "Abel'sche Functionen von drei Variabeln," §1; Baker, "Abel's Theorem," etc., Art. 189.

† These are, in matrix form, two of the well-known bilinear relations between the periods (15); see Baker, Art. 140.

‡ Compare for the notation end of §1.

a). *The partial derivatives of Θ with respect to $a_{\alpha\beta}$, $b_{\alpha\beta}$, $\tau_{\alpha\beta}$*: By differentiating the series (18) on the one hand with respect to the quantities $a_{\alpha\beta}$, $b_{\alpha\beta}$, $\tau_{\alpha\beta}$, on the other hand, with respect to the u_α's, the following theorem is easily verified:

Theorem III.

Considered as a function of the quantities

$$u_\alpha, \; a_{\alpha\beta}, \; b_{\alpha\beta}, \; \tau_{\alpha\beta},$$

Weierstrass' function

$$\Theta(u_1, u_2, \ldots, u_\rho;\, m, n)$$

satisfies the following partial differential equations:

$$\left.\begin{aligned}
&\frac{\partial\Theta}{\partial a_{\alpha\alpha}} = u_\alpha^2 . \Theta, \quad \frac{\partial\Theta}{\partial a_{\alpha\beta}} = 2u_\alpha u_\beta . \Theta, \qquad\qquad \beta \neq \alpha \\
&u_\beta \frac{\partial\Theta}{\partial u_\gamma} = 2\Theta . \sum_\alpha a_{\alpha\gamma} u_\alpha u_\beta + \sum_\alpha b_{\alpha\gamma} \frac{\partial\Theta}{\partial b_{\alpha\beta}} \\
&\frac{\partial^2\Theta}{\partial u_\gamma \partial u_\delta} = 2\Theta . \Big[a_{\gamma\delta} + 2\sum_{\alpha,\beta} a_{\alpha\gamma} a_{\beta\delta} u_\alpha u_\beta\Big] + 2\sum_{\alpha,\beta} a_{\gamma\alpha} b_{\beta\delta} \frac{\partial\Theta}{\partial b_{\beta\alpha}} \\
&\qquad + 2\sum_{\alpha,\beta} a_{\delta\beta} b_{\alpha\gamma} \frac{\partial\Theta}{\partial b_{\alpha\beta}} + 2\pi i \Big[2\sum_\alpha b_{\alpha\gamma} b_{\alpha\delta} \frac{\partial\Theta}{\partial \tau_{\alpha\alpha}} + {\sum_{\alpha,\beta}}' b_{\alpha\gamma} b_{\beta\delta} \frac{\partial\Theta}{\partial \tau_{\alpha\beta}}\Big],
\end{aligned}\right\} \quad (\mathrm{C})$$

where ${\sum\limits_{\alpha,\beta}}'$ *indicates that in the summation the terms for which* $\beta = \alpha$ *are to be omitted.*

The three systems of equations contained in (C) can again be written in the form of three matrix equations. Let

$$m_{\alpha\beta} \begin{cases} \dfrac{\partial\Theta}{\partial a_{\alpha\beta}} & \beta \neq \alpha \\ 2\dfrac{\partial\Theta}{\partial a_{\alpha\alpha}} & \beta = \alpha \end{cases}; \qquad n_{\alpha\beta} = \begin{cases} \dfrac{\partial\Theta}{\partial \tau_{\alpha\beta}} & \beta \neq \alpha \\ 2\dfrac{\partial\Theta}{\partial \tau_{\alpha\alpha}} & \beta = \alpha \end{cases}, \qquad (20)$$

$$P = \left(\left|\frac{\partial\Theta}{\partial b_{\alpha\beta}}\right|\right), \quad U = \left(\left|u_\alpha u_\beta\right|\right), \quad V = \left(\left|u_\alpha \frac{\partial\Theta}{\partial u_\beta}\right|\right), \quad W = \left(\left|\frac{\partial^2\Theta}{\partial u_\alpha \partial u_\beta}\right|\right),$$

and observe that $\overline{M} = M$, $\overline{N} = N$, $\overline{U} = U$, $\overline{W} = W$. With these notations the three matrix equations replacing (C) are:

$$\left.\begin{aligned}
M &= 2\Theta . U, \\
V &= 2\Theta . UA + \overline{P}B, \\
W &= 2\Theta . (A - 2AUA) + 2(AV + \overline{V}A) + 2\pi i\, \overline{B}NB.
\end{aligned}\right\} \quad (\mathrm{C}')$$

In these equations Θ plays the part of a scalar factor; they can be immediately solved with respect to N and $\overline{P}$, and in order to obtain P we have only to remember that for any two matrices A and B the rules hold:

$$\overline{A+B} = \bar{A} + \bar{B}$$

and

$$\overline{AB} = \bar{B}\bar{A}.$$

b). *The derivatives of $a_{\alpha\beta}$, $b_{\alpha\beta}$, $\tau_{\alpha\beta}$ with respect to a* can be found by combining (A′) and (B′) with (16). The matrix equation

$$2A\Omega = H,$$

written non-symbolically, reads

$$2\sum_{\beta} a_{\alpha\beta}\,\omega_{\beta\gamma} = \eta_{\alpha\gamma},$$

hence

$$2\sum_{\beta} \frac{\partial a_{\alpha\beta}}{\partial a}\,\omega_{\beta\gamma} + 2\sum_{\beta} a_{\alpha\beta}\,\frac{\partial \omega_{\beta\gamma}}{\partial a} = \frac{\partial \eta_{\alpha\gamma}}{\partial a},$$

and if we denote again

$$\delta A = \left(\frac{\partial a_{\alpha\beta}}{\partial a}\right), \quad \delta B = \left(\frac{\partial b_{\alpha\beta}}{\partial a}\right), \quad \delta T = \left(\frac{\partial \tau_{\alpha\beta}}{\partial a}\right),$$

the last equation may be written in matrix form

$$2\delta A\,.\,\Omega + 2A\,.\,\delta\Omega = \delta H,$$

similarly

$$\begin{aligned} \delta B\,.\,\Omega + B\,.\,\delta\Omega &= 0, \\ \delta\Omega\,.\,T + \Omega\,.\,\delta T &= \delta\Omega'. \end{aligned}$$

Substituting for $\delta\Omega$, $\delta\Omega'$, δH their values from (A′), (B′) and solving for δA, δB, δT, we obtain the

Theorem IV.

The expressions for the derivatives of the quantities $a_{\alpha\beta}$, $b_{\alpha\beta}$, $\tau_{\alpha\beta}$ with respect to a branch-point a are exhibited in the following matrix equations:

$$\left.\begin{aligned} \delta A &= -\tfrac{1}{2}\Lambda - (KA + A\bar{K}) + 2A\Gamma A, \\ \delta B &= -B(\bar{K} - 2\Gamma A), \\ \delta T &= 2\pi i\,B\Gamma\bar{B}. \end{aligned}\right\} \tag{D}$$

c). We now return to equation (19). Using the notations (20), we may write it

$$\frac{\partial\Theta}{\partial a} = \tfrac{1}{2}\sum_{\alpha,\beta} m_{\alpha\beta}\frac{\partial a_{\alpha\beta}}{\partial a} + \sum_{\alpha,\beta}\frac{\partial\Theta}{\partial b_{\alpha\beta}}\,\frac{\partial b_{\alpha\beta}}{\partial a} + \tfrac{1}{2}\sum_{\alpha,\beta} n_{\alpha\beta}\frac{\partial \tau_{\alpha\beta}}{\partial a}.$$

This expression can be made accessible to the methods of the theory of matrices by the following remark:

Let, for any square matrix A of ρ^2 elements, $\{A\}$ denote the *sum of the elements in the principal diagonal*,* that is

$$\{A\} = \sum_{\alpha} a_{\alpha\alpha}.$$

From this definition of the symbol $\{A\}$ follow at once the following rules:

$$\left.\begin{aligned} \{\overline{A}\} &= \{A\}, \\ \{A+B\} &= \{A\}+\{B\}, \\ \{AB\} &= \sum_{\alpha,\beta} a_{\alpha\beta}\, b_{\beta\alpha} = \{BA\}. \\ &\text{From the last equation follows:} \\ \{ABC\} &= \{BCA\} = \{CAB\} \\ \text{and} \quad \sum_{\alpha,\beta} a_{\alpha\beta}\, b_{\alpha\beta} &= \{A\overline{B}\} = \{\overline{A}B\}. \end{aligned}\right\} \tag{21}$$

Our expression for $\frac{\partial\Theta}{\partial a}$ may therefore be written

$$\frac{\partial\Theta}{\partial a} = \{\tfrac{1}{2}\overline{M}.\delta A + \overline{P}.\delta B + \tfrac{1}{2}\overline{N}.\delta T\}.$$

If we substitute for M, N, P and δA, δB, δT their values from (C), (C′), (D), we obtain

$$\begin{aligned} \frac{\partial\Theta}{\partial a} = \{ &-\tfrac{1}{2}\Theta.U\Lambda - \Theta.A\Gamma + \Theta(UA\overline{K} - UKA) \\ &+ 2\Theta(AUA\Gamma - UA\Gamma A) + (2V\Gamma A - AV\Gamma - \overline{V}A\Gamma) + \tfrac{1}{2}W\Gamma - V\overline{K}\}. \end{aligned}$$

This expression simplifies considerably, if we apply the rules (21), and remember that $\overline{A} = A$, $\overline{U} = \overline{U}$, $\Gamma = \Gamma$:

$$\left.\begin{aligned} \{UA\overline{K}\} &= \{K\overline{A}\overline{U}\} = \{KAU\} = \{UKA\}, \\ \{AUA\Gamma\} &= \{UA\Gamma A\} = \{\Gamma AUA\}, \\ \{V\Gamma A\} &= \{AV\Gamma\} = \Gamma AV\} = \{\overline{V}\overline{A}\overline{\Gamma}\} = \{\overline{V}A\Gamma\}. \end{aligned}\right\} \tag{22}$$

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* Since the above was written, I found that the operation $\{A\}$ has been studied by Taber, in his paper "On the Application to Matrices of any Order of the Quaternion Symbols S and V," Proc. London Math. Soc., vol. XXII, p. 67.

Thus $\frac{\partial\Theta}{\partial a}$ reduces to

$$\frac{\partial\Theta}{\partial a}=\{\tfrac{1}{2}\Theta\,.\,U\Lambda-\Theta\,.\,A\Gamma-V\overline{K}+\tfrac{1}{2}W\Gamma\}, \tag{23}$$

and returning to non-symbolical notation we have the

Theorem V.

Weierstrass' most general Θ-function defined by (18) *satisfies the partial differential equation*

$$\frac{\partial\Theta}{\partial a}=-\tfrac{1}{2}\Theta\,.\Big[\sum_{\alpha,\,\beta}\lambda_{\alpha\beta}u_\alpha u_\beta+4\sum_{\alpha,\,\beta}\frac{g_\alpha(a)\,g_\beta(a)\,a_{\alpha\beta}}{R'(a)}\Big] -\sum_{\alpha,\,\beta}\varkappa_{\alpha\beta}u_\alpha\frac{\partial\Theta}{\partial u_\beta}+\sum_{\alpha,\,\beta}\frac{g_\alpha(a)\,g_\beta(a)}{R'(a)}\,\frac{\partial^2\Theta}{\partial u_\alpha\,\partial u_\beta}, \tag{E}$$

the coefficients $\varkappa_{\alpha\beta}$ *and* $\lambda_{\alpha\beta}$ *being defined by* (7) *and* (12).

§5.—*The Partial Differential Equations for the Θ-Functions: Second Proof.*

According to Weierstrass, the function $\Theta(u_1\ldots\ldots u_\rho;\ m,\,n)$ is connected with the function

$$\vartheta(v_1\ldots\ldots v_\rho;\ m,\,n)=\sum_{\nu_1,\nu_2\ldots\ldots\nu_\rho}e^{\pi i\sum\limits_{\alpha,\beta}\tau_{\alpha\beta}(\nu_\alpha+n_\alpha)(\nu_\beta+n_\beta)+2\pi i\sum\limits_{\alpha}(\nu_\alpha+n_\alpha)(v_\alpha+m_\alpha)} \tag{24}$$

by the relation

$$\Theta(u_1\ldots\ldots u_\rho;\ m,\,n)=e^{g(u_1,\ldots\ldots,u_\rho)}\,\vartheta(v_1,\,v_2,\,\ldots\ldots,\,v_\rho;\ m,\,n), \tag{25}$$

where

$$g(u_1\ldots\ldots u_\rho)=\sum_{\alpha,\,\beta}a_{\alpha\beta}u_\alpha u_\beta, \tag{26}$$

and the v_β's are determined by

$$u_\alpha=\sum_\beta 2\omega_{\alpha\beta}v_\beta. \tag{27}$$

The function ϑ satisfies the partial differential equations

$$\frac{\partial^2\vartheta}{\partial v_\alpha^2}=4\pi i\,\frac{\partial\vartheta}{\partial\tau_{\alpha\alpha}},\quad \frac{\partial^2\vartheta}{\partial v_\alpha\,\partial v_\beta}=2\pi i\,\frac{\partial\vartheta}{\partial\tau_{\alpha\beta}}.\qquad(\beta\neq\alpha) \tag{28}$$

Hence

$$\frac{\partial\Theta}{\partial a}=e^{g(u_1\ldots\ldots u_\rho)}\,\frac{\partial\vartheta}{\partial a}+\Theta\,.\,\frac{\partial g(u_1\ldots\ldots u_\rho)}{\partial a}.$$

a). Since a is implicitly contained in the v_α's as well as in the $\tau_{\alpha\beta}$'s, we have

$$\frac{\partial\vartheta}{\partial a}=\sum_{\alpha}\frac{\partial\vartheta}{\partial a_{\alpha\alpha}}\,\frac{\partial\tau_{\alpha\alpha}}{\partial a}+\sum_{(\alpha,\,\beta)}\frac{\partial\vartheta}{\partial\tau_{\alpha\beta}}\,\frac{\partial\tau_{\alpha\beta}}{\partial a}+\sum_{\alpha}\frac{\partial\vartheta}{\partial v_\alpha}\,\frac{\partial v_\alpha}{\partial a},$$

or on account of (28),

$$\frac{\partial\vartheta}{\partial a}=\frac{1}{4\pi i}\sum_{\alpha,\,\beta}\frac{\partial^2\vartheta}{\partial v_\alpha\,\partial v_\beta}\,\frac{\partial\tau_{\alpha\beta}}{\partial a}+\sum_{\alpha}\frac{\partial\vartheta}{\partial v_\alpha}\,\frac{\partial v_\alpha}{\partial a}.$$

The first term on the right-hand side may be written

$$\frac{1}{4\pi i}\left\{\delta T.\left(\frac{\partial^2\vartheta}{\partial v_\alpha\,\partial v_\beta}\right)\right\}.$$

But from (27) follows

$$\left(\frac{\partial^2\vartheta}{\partial v_\alpha\,\partial v_\beta}\right)=4\overline{\Omega}.\left(\frac{\partial^2\vartheta}{\partial u_\alpha\,\partial u_\beta}\right).\Omega.$$

Substituting this value and the value for δT from (D), and making use of (16) and (21), we obtain

$$\frac{1}{4\pi i}\sum_{\alpha,\,\beta}\frac{\partial^2\vartheta}{\partial v_\alpha\,\partial v_\beta}\,\frac{\partial\tau_{\alpha\beta}}{\partial a}=\left\{\tfrac{1}{2}\left(\frac{\partial^2\vartheta}{\partial u_\alpha\,\partial u_\beta}\right).\Gamma\right\}. \tag{29}$$

b). On the other hand

$$\sum_{\alpha}\frac{\partial\vartheta}{\partial v_\alpha}\,\frac{\partial v_\alpha}{\partial a}=\sum_{\alpha,\,\beta}\frac{\partial\vartheta}{\partial u_\beta}\,2\omega_{\beta\alpha}\,\frac{\partial v_\alpha}{\partial a},$$

but

$$\frac{\partial u_\beta}{\partial a}=0=\sum_{\alpha}2\omega_{\beta\alpha}\,\frac{\partial v_\alpha}{\partial a}+\sum_{\alpha}2\,\frac{\partial\omega_{\beta\alpha}}{\partial a}\,v_\alpha,$$

$$\therefore\quad\sum_{\alpha}\frac{\partial\vartheta}{\partial v_\alpha}\,\frac{\partial v_\alpha}{\partial a}=-2\sum_{\alpha,\,\beta}v_\alpha\,\frac{\partial\vartheta}{\partial u_\beta}\cdot\frac{\partial\omega_{\beta\alpha}}{\partial a}=-2\left\{\delta\Omega.\left(v_\alpha\,\frac{\partial\vartheta}{\partial u_\beta}\right)\right\}.$$

From (27) follows

$$\left(v_\alpha\,\frac{\partial\vartheta}{\partial v_\beta}\right)=\tfrac{1}{2}\,\Omega^{-1}\cdot\left(u_\alpha\,\frac{\partial\vartheta}{\partial u_\beta}\right);$$

hence if we substitute for $\delta\Omega$ its value from (A′) and remember that $H\Omega^{-1}=2A$, we obtain

$$\sum_{\alpha}\frac{\partial\vartheta}{\partial v_\alpha}\,\frac{\partial v_\alpha}{\partial a}=\left\{(-\overline{K}+2\Gamma A).\left(u_\alpha\,\frac{\partial\vartheta}{\partial u_\beta}\right)\right\}. \tag{30}$$

c). Now from (25)

$$e^{g(u_1 \,\ldots .\, u_\rho)} \frac{\partial^2 \vartheta}{\partial u_\alpha \partial u_\beta} = \frac{\partial^2 \Theta}{\partial u_\alpha \partial u_\beta} - \frac{\partial \Theta}{\partial u_\alpha} \frac{\partial g}{\partial u_\beta} - \frac{\partial \Theta}{\partial u_\beta} \frac{\partial g}{\partial u_\alpha} - \Theta \cdot \frac{\partial^2 g}{\partial u_\alpha \partial u_\beta} + \Theta \cdot \frac{\partial g}{\partial u_\alpha} \frac{\partial g}{\partial u_\beta}.$$

But from the definition of $g(u_1 \ldots . u_\rho)$ follows

$$\left\{ \frac{\partial \Theta}{\partial u_\alpha} \frac{\partial g}{\partial u_\beta} \right\} = 2\overline{V}A, \quad \left\{ \frac{\partial g}{\partial u_\alpha} \frac{\partial \Theta}{\partial u_\beta} \right\} = 2AV,$$

$$\left\{ \frac{\partial^2 g}{\partial u_\alpha \partial u_\beta} \right\} = 2A, \quad \left\{ \frac{\partial g}{\partial u_\alpha} \frac{\partial g}{\partial u_\beta} \right\} = 4AUA.$$

Further,

$$e^{g(u_1 \,\ldots .\, u_\rho)} u_\alpha \frac{\partial \vartheta}{\partial u_\beta} = u_\alpha \frac{\partial \Theta}{\partial u_\beta} - \Theta \,.\, u_\alpha \frac{\partial g}{\partial u_\beta},$$

and

$$\left\{ u_\alpha \frac{\partial g}{\partial u_\beta} \right\} = 2UA.$$

Substituting the values thus obtained for $\left\{ \frac{\partial^2 \vartheta}{\partial u_\alpha \partial u_\beta} \right\}$ and $\left\{ u_\alpha \frac{\partial \vartheta}{\partial u_\beta} \right\}$ in (29) and (30) and making use of the relations (22), we reach the result

$$e^{g(u_1 \,\ldots .\, u_\rho)} \frac{\partial \vartheta}{\partial a} = \{ -\overline{K}V + 2\Theta \,.\, \overline{K}UA - 2\Theta \,.\, AUA\Gamma - \Theta \,.\, A\Gamma + \tfrac{1}{2} W\Gamma \}. \qquad (31)$$

d). Finally,

$$\frac{\partial g}{\partial a} = \sum_{\alpha,\beta} \frac{\partial a_{\alpha\beta}}{\partial a} u_\alpha u_\beta = \{ \delta A \,.\, U \},$$

therefore, replacing δA by its values from (D), we have

$$\Theta \cdot \frac{\partial g}{\partial a} = \Theta \,.\, \{ -\tfrac{1}{2} \Lambda U - KAU - A\overline{K}U + 2A\Gamma AU \}. \qquad (32)$$

Adding (31) and (32) and observing that

$$\{ \overline{K}UA \} = AUK \} = \{ KAU \} = \{ UKA \} = \{ A\overline{K}U \},$$

we reach the final result

$$\frac{\partial \Theta}{\partial a} = \{ -\tfrac{1}{2} \Theta \,.\, \Lambda U - \Theta \,.\, A\Gamma - \overline{K}V + \tfrac{1}{2} W\Gamma \},$$

in accordance with (23).

§6.—*The Partial Differential Equation for Wiltheiss' Function* $Th(u_1 \ldots . u_\rho)$.

If ω denote the determinant

$$|\omega_{\alpha\beta}| = \omega ,$$

and $Adj\omega_{\alpha\beta}$ the minor of $\omega_{\alpha\beta}$ in this determinant, we have

$$\frac{\partial \omega}{\partial a} = \sum_{\alpha, \beta} \frac{\partial \omega_{\alpha\beta}}{\partial a} Adj\omega_{\alpha\beta} = \{\delta\Omega \,.\, \{Adj\omega_{\alpha\beta}\}\}.$$

But

$$\left(\frac{\overline{Adj\omega_{\alpha\beta}}}{\omega}\right) = \Omega^{-1},$$

therefore

$$\frac{\partial \log \omega}{\partial a} = \{\delta\Omega \,.\, \Omega^{-1}\} = \{\overline{K} - \Gamma H \Omega^{-1}\},$$

or

$$\frac{\partial \log \omega}{\partial a} = \{K\} - 2\{\Gamma A\}. \tag{33}$$

To obtain the value of $\{K\}$, observe first that $\{K\}$ remains invariant, if we pass from one set of integrals of the first kind, w_α, to another, $\bar{w}_\alpha$, the function $F(x, \xi)$ remaining the same.

For if*

$$g_\alpha(x) = \sum_\gamma c_{\alpha\gamma} \bar{g}_\alpha(x),$$

we have at the same time,

$$\overline{h}_\delta(\xi) = \sum_\beta c_{\beta\delta} h_\beta(\xi),$$

and if we write

$$\sum_{\alpha, \beta} \varkappa_{\alpha\beta} g_\alpha(x) h_\beta \xi = \sum_{\gamma, \delta} \overline{\varkappa}_{\gamma\delta} \bar{g}_\gamma(x) \overline{h}_\delta(\xi),$$

we obtain

$$\overline{K} = \bar{C} K \bar{C}^{-1}, \tag{34}$$

and therefore according to the rules (21),

$$\{\overline{K}\} = \{K\}. \tag{35}$$

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* The $\bar{g}_\alpha^{(x)}(x)$'s are again supposed independent of a

Further, it follows from (7), if we make use of (6), that

$$\sum_{\alpha} \varkappa_{\alpha\beta}\, g_\alpha(x) = \tfrac{1}{2}\,\frac{g_\beta(x)}{x-a} - \frac{g_\beta(a)\,F(x,\,a)}{R'(a)\,(x-a)}$$

or since by (5),

$$\frac{F(x,\,a)}{(x-a)^2} = \tfrac{1}{2}\,\frac{R'(a)}{x-a} - 2\sum_{\alpha} g_{\rho+\alpha}(a)\,g_\alpha(x), \tag{36}$$

$$\sum_{\alpha} \varkappa_{\alpha\beta}\, g_\alpha(x) = \tfrac{1}{2}\,\frac{g_\beta(x) - g_\beta(a)}{x-a} + \frac{2g_\beta(a)}{R'(a)}\sum_{\alpha} g_{\rho+\alpha}(a)\,g_\alpha(x).$$

And if we choose, as we may according to the above remark, $g_\alpha(x) = x^{\alpha-1}$, we obtain

$$\varkappa_{\alpha\alpha} = \frac{2g_\alpha(a)\,g_{\rho+\alpha}(a)}{R'(a)},$$

therefore

$$\{K\} = \sum_{\alpha} \varkappa_{\alpha\alpha} = \frac{2}{R'(a)}\sum_{\alpha} g_\alpha(a)\,g_{\rho+\alpha}(a),$$

or if we put in (36) $x = a$:

$$\{K\} = -\frac{1}{2R'(a)}\left(\frac{\partial^2 F(x,\,a)}{\partial x^2}\right)_{x=a}. \tag{37}$$

So far the function $F(x,\,\xi)$ was only subject to the conditions (2). We now introduce the special assumption that $F(x,\,\xi)$ *shall be the* $\rho + 1^{st}$ *polar of* $R(x)$ *with respect to* ξ (Klein); then

$$\{K\} = -\frac{\rho}{4\,(2\rho+1)}\,\frac{R''(a)}{R'(a)}, \tag{38}$$

$R''(x)$ denoting the second derivative of $R(x)$.

We thus reach the

Theorem VI.

Under the assumption that $F(x,\,\xi)$ *is the* $\rho + 1^{st}$ *polar of* $R(x)$ *with respect to* ξ, *the logarithmic derivative of the determinant* ω *with respect to* a *is expressible as follows:*

$$\frac{\partial \log \omega}{\partial a} = -\frac{\rho}{4\,(2\rho+1)}\,\frac{R''(a)}{R'(a)} - 4\sum_{\alpha,\,\beta}\frac{g_\alpha(a)\,g_\beta(a)\,a_{\alpha\beta}}{R'(a)}. \tag{F}$$

By combining this result with (E) we immediately obtain

Theorem VII.

Under the same assumption, Wiltheiss' function

$$Th\,(u_1\,\ldots\,u_\rho) = \left(\frac{\pi}{2}\right)^{\frac{\rho}{2}} \omega^{-\frac{1}{2}}\,\Theta\,(u_1\,\ldots\,u_\rho)$$

satisfies the differential equation

$$\frac{\partial\, Th}{\partial a} = -\frac{\rho}{8\,(2\rho+1)}\,\frac{R''(a)}{R'(a)} - \tfrac{1}{2}\, Th\,.\sum_{\alpha,\,\beta} \lambda_{\alpha\beta}\, u_\alpha\, u_\beta$$
$$-\sum_{\alpha,\,\beta} \varkappa_{\alpha\beta}\, u_\alpha\, \frac{\partial\, Th}{\partial u_\beta} + \frac{1}{R'(a)} \sum_{\alpha,\,\beta} \frac{\partial^2 Th}{\partial u_\alpha\, \partial u_\beta}\, g_\alpha(a)\, g_\beta(a)\,. \qquad \text{(G)}$$

§7.—*The Partial Differential Equations for the Functions* $\sigma_{\phi_{\rho+1}\psi_{\rho+1}}$.

Suppose, as in the previous §, $F(x,\,\xi)$ to be the $\rho+1^{\text{st}}$ polar of $R(x)$ with respect to ξ, and let

$$R(x) = \phi(x)\,.\,\psi(x)$$

be a decomposition of $R(x)$ into two factors of degree $\rho+1$, $\Theta_{\phi\psi}$ and $\sigma_{\phi\psi}$ the corresponding Θ- and σ-functions. Then $\Theta_{\phi\psi}(0,\,0,\,\ldots\,,\,0) \neq 0$ and

$$\sigma_{\phi\psi}(u_1\,\ldots\,u_\rho) = \frac{\Theta_{\phi\psi}(u_1\,\ldots\,u_\rho)}{\Theta_{\phi\psi}(0\,\ldots\,0)}\,. \qquad (39)$$

If in (E) we put

$$u_1 = 0\,, \quad u_2 = 0\,\ldots\,u_\rho = 0\,,$$

we obtain

$$\frac{\partial \log \Theta(0_1\,\ldots\,0)}{\partial a} = -\,2\sum_{\alpha,\,\beta} \frac{g_\alpha(a)\, g_\beta(a)\, a_{\alpha\beta}}{R'(a)} + \sum_{\alpha,\,\beta} \frac{\partial^2 \log \Theta}{\partial u_\alpha\, \partial u_\beta}\Big)_0 \frac{g_\alpha(a)\, g_\beta(a)}{R'(a)}\,. \qquad (40)$$

But for arbitrary parameters s, t[*]

$$\sum_{\alpha,\,\beta} \left(\frac{\partial^2 \log \Theta}{\partial u_\alpha\, \partial u_\beta}\right)_0 g_\alpha(s)\, g_\beta(t) = \frac{\phi(s)\,\psi(t) + \phi(t)\,\psi(s) - 2F(s,\,t)}{4\,(t-s)}\,. \qquad (41)$$

Expand $\phi(s)\,\psi(t) + \phi(t)\,\psi(s)$ into Clebsch-Gordan's series, and put $t = s$:

$$\sum_{\alpha,\,\beta} \left(\frac{\partial^2 \log \Theta}{\partial u_\alpha\, \partial u_\beta}\right)_0 g_\alpha(s)\, g_\beta(s) = \frac{\rho\,(\rho+1)^2}{8\,(2\rho+1)}\,(\phi,\,\psi)_2, \qquad (42)$$

$(\phi,\,\psi)_2$ denoting the second transvectant of $\phi(s)$ and $\psi(s)$.

Putting $s = a$ and combining (40) and (E), we obtain the

Theorem VIII.

The σ-function which belongs to the decomposition of $R(x)$ into the two factors of degree $\rho+1$:

$$R(x) = \phi(x)\,\psi(x)\,,$$

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[*] See Bolza, American Journal of Mathematics, vol. XVI, p. 80.

satisfies the partial differential equation

$$\frac{\partial \sigma_{\phi\psi}}{\partial a} = -\tfrac{1}{2}\sigma_{\phi\psi}\cdot\sum_{\alpha,\beta}\lambda_{\alpha\beta}\,u_\alpha u_\beta - \frac{\rho(\rho+1)^2}{8(2\rho+1)}\,\frac{f(a)}{R'(a)}\,\sigma_{\phi\psi}$$
$$-\sum_{\alpha,\beta}\varkappa_{\alpha\beta}\,u_\alpha\,\frac{\partial\sigma_{\phi\psi}}{\partial u_\beta} + \sum_{\alpha,\beta}\frac{\partial^2\sigma_{\phi\psi}}{\partial u_\alpha\,\partial u_\beta}\,\frac{g_\alpha(a)\,g_\beta(a)}{R'(a)}, \qquad \text{(H)}$$

where $f(x) = (\phi,\psi)_2 = (\phi\psi)^2\phi_x^{\rho-1}\psi_x^{\rho-1}$.

From (40) it is easy to derive Thomae's* expression for $d\log\Theta_{\phi\psi}(0\,\ldots\,.\,0)$ in terms of the branch-points. In fact, if a be a root of the factor $\phi(x)$, we obtain from (41) by first letting $s = a$ and then $t = a$:

$$\sum_{\alpha,\beta}\left(\frac{\partial^2\log\Theta}{\partial u_\alpha\,\partial u_\beta}\right)_0 g_\alpha(a)\,g_\beta(a) = \tfrac{1}{8}\,\psi(a)\,\phi''(a) - \frac{\rho}{8(2\rho+1)}\,R''(a),$$

and combining this result with (40) and (F), we obtain

$$\frac{\partial}{\partial a}\log\frac{\Theta(0\,\ldots\,.\,0)}{\omega^{\frac{1}{2}}} = \tfrac{1}{8}\,\frac{\phi''(a)}{\phi'(a)} = \tfrac{1}{8}\,\frac{\partial\log\Delta_\phi}{\partial a}, \qquad (43)$$

Δ_ϕ being the discriminant of ϕ.

Similarly, if b be a root of $\psi(x)$, we have

$$\frac{\partial}{\partial b}\log\frac{\Theta(0_1\,\ldots\,.\,0)}{\omega^{\frac{1}{2}}} = \tfrac{1}{8}\,\frac{\psi''(b)}{\psi'(b)} = \tfrac{1}{8}\,\frac{\partial\log\Delta_\psi}{\partial b}.$$

Hence
$$d\log\Theta(0\,\ldots\,.\,0) = d\log\omega^{\frac{1}{2}}\,\Delta_\phi^{\frac{1}{8}}\,\Delta_\psi^{\frac{1}{8}}, \qquad (44)$$

which is Thomae's result.

UNIVERSITY OF CHICAGO, *September 22d*, 1898.

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[*] Crelle's Journal, Bd. 71. p. 201; compare also Schroeder, "Ueber den Zusammenhang der Hyperelliptischen Ϭ- und ϑ-Functionen," §12–§15.

On Certain Differential Equations of the Second Order Allied to Hermite's Equation.

By Edward B. Van Vleck.

Hermite's differential equation

$$\frac{d^2y}{du^2} = [n(n+1)p(u) + h]\,y$$

can be thrown by the substitution

$$x = p(u) \text{ or } u = \int \frac{dx}{2\sqrt{f(x) = (x-e_1)(x-e_2)(x-e_3)}}$$

into the form

$$f(x)\frac{d^2y}{dx^2} + \frac{f'(x)}{2}\,\frac{dy}{dx} - \frac{n(n+1)x+h}{4}\,y = 0.$$

As is well known, it admits of two solutions whose product is a polynomial in x. Other differential equations of the second order which have the same or an analogous property have been given by Fuchs,[*] Brioschi,[†] Markoff,[‡] Lindemann,[§] and G. W. Hill.[||] Markoff confines his attention to the hypergeometric equation, Fuchs and Brioschi to differential equations in which the coefficient of $\frac{dy}{dx}$ is one-half the derivative of the coefficient of $\frac{d^2y}{dx^2}$. Lindemann, in his discussion of the "differential equation of the functions of the elliptic cylinder," a limiting form of Hermite's equation, proves that it admits of two solutions whose product is a holomorphic function. Hill's equation is an extension of this equation, and possesses the same property.

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[*] Annali di Matematica, Ser. II, t. IX.

[†] Annali di Matematica, Ser. II, t. IX, p. 11.

[‡] Math. Ann., Bd. 28.

[§] Math. Ann., Bd. 22.

[||] Acta Mathematica, Bd. 8.

The object of the first section of this paper is to determine in general what regular differential equations of the second order admit of two solutions whose product is a polynomial. It will be found that there are several distinct classes of such equations under which those hitherto considered are comprised as special cases. Incidentally we shall obtain a class of irregular equations with three singular points, which includes the equations of Lindemann and of Hill.

The properties of the two solutions and of their quotient η will be developed in the second section. In particular, it will be shown that the monodromic group of substitutions of η can be thrown into the form

$$\bar{\eta} = \frac{\alpha}{\eta}, \quad \bar{\eta} = \beta\eta,$$

and that, conversely, if the group of any regular differential equation can be thus expressed, there will be two solutions whose product is a polynomial multiplied by certain factors which correspond to the singular points and can be removed by an elementary substitution. So far as I am aware, the identity of these two classes of equations has not been hitherto noted. The other properties developed are for the most part extensions of properties given by Hermite and Klein for Hermite's equation, but to effect the generalization a new method is employed which is independent of elliptic integrals. The third section of the paper is devoted chiefly to an investigation of the position of the real roots of the polynomial product with reference to the singular points, when these points are real and their number is limited to four. Klein's investigation[*] for Hermite's equation here also paves the way, but the "Oscillation theorem" upon which it is based is inadequate to the more general discussion, and recourse is had to the method of conformal representation.

I.

§1. Any regular linear differential equation of the second order with a singular point at ∞ may be written in the form

$$\frac{d^2y}{dx^2} + \sum_{i=1}^{i=r} \left(\frac{1-\lambda_i'-\lambda_i''}{x-e_i}\right) \frac{dy}{dx}$$
$$+ \left(\frac{\lambda_\infty'\lambda_\infty''-\Sigma\lambda_i'\lambda_i''x^{r-2}+a_1x^{r-3}+\dots+a_{r-2}}{\Pi(x-e_i)} + \sum \frac{\lambda_i'\lambda_i''}{(x-e_i)^2}\right) y = 0, \quad [1]$$

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[*] Math. Ann., Bd. 40.

where
$$\Sigma(\lambda_i' + \lambda_i'') + \lambda_\infty' + \lambda_\infty'' = r - 1. \qquad [2]$$

The singular points e_i will here be supposed to be given, but the "*accessory parameters*" $a_1, \ldots., a_{r-2}$ and the exponents λ_i', λ_i'' are to be so determined that the product of two particular integrals shall be a polynomial P_n of the n^{th} degree. The two fundamental integrals for e_i have in general the form

$$\left.\begin{array}{l} P^{\lambda_i} = [\pm (x - e_i)^{\lambda_i'}][1 + B(x - e_i) + C(x - e_i)^2 + \ldots.] \\ P^{\lambda_i''} = [\pm (x - e_i)^{\lambda_i''}][1 + B'(x - e_i) + C'(x - e_i)^2 + \ldots.] \end{array}\right\}, \qquad [3]$$

the leading coefficient in each series for convenience being taken equal to unity. When, however, the difference of the two exponents is an integer, one of these integrals must in general be modified by the introduction of a logarithmic term. In the first factor of each expansion a definite sign is to be attached to the binomial, but for the present it is immaterial which sign is selected. The corresponding expansions for the singular point ∞ are

$$P^{\lambda_\infty'} = \left[\pm\left(\frac{1}{x}\right)^{\lambda_\infty'}\right]\left[1 + \frac{B}{x} + \frac{C}{x^2} + \ldots.\right]$$

and a similar series for P^{λ} .

The foregoing expansions hold only over a limited portion of the x-plane. When, however, the product of two solutions is a polynomial, the integration of the equation can be effected by familiar methods, and its general integral will be expressed in terms of two particular integrals which hold over the entire plane. Two cases are possible, according as the two solutions forming the polynomial product are identical or distinct. In either case the polynomial itself satisfies the differential equation

$$\frac{d^3y}{dx^3} + 3p\frac{d^2y}{dx^2} + \left(\frac{dp}{dx} + 2p^2 + 4q\right)\frac{dy}{dx} + \left(4pq + 2\frac{dq}{dx}\right)y = 0,$$

where p and q denote the coefficients of $\frac{dy}{dx}$ and of y in [1], and it is obtained by substituting for y in this equation a polynomial of the n^{th} degree with unknown coefficients. When the two solutions are identical, their common value y_1 is the square root of the polynomial. A second integral can be obtained by means of the well-known relation

$$y_1 y_2' - y_1' y_2 = C \prod_{i=1}^{i=r} (x - e_i)^{\lambda_i' + \lambda_i'' - 1}, \qquad [4]$$

which exists between any two independent integrals of the equation. This gives for the quotient of the two integrals

$$\eta = \frac{y_2}{y_1} = \int \frac{C\,dx}{y_1^2 \Pi (x - e_i)^{1 - \lambda_i' - \lambda_i''}}. \qquad [5]$$

In the second case, if y_1, y_2 represent the distinct solutions, differentiating the equation $y_1 y_2 = P_n$ and combining with [3], we find

$$\left.\begin{aligned} y_1 &= C' \sqrt{P_n}\, e^{\sigma \int \frac{dx}{P_n . \Pi (x - e_i)^{1 - \lambda_i' - \lambda_i''}}} \\ y_2 &= C'' \sqrt{P_n}\, e^{-\sigma \int \frac{dx}{P_n . \Pi (x - e_i)^{1 - \lambda_i' - \lambda_i''}}} \end{aligned}\right\}, \qquad [6]$$

where C, C' and C'' are constants. These formulæ hold equally well when for P_n a holomorphic function can be substituted.

§2. We proceed now to determine the conditions under which the square of a single solution y_1 can be a polynomial of the n^{th} degree. Let y_1 at any singular point in the finite plane be expressed as $aP^{\lambda_i'} + bP^{\lambda_i''}$. Since the expansion of its square into a series is to begin either with a constant or with a positive integral power of $x - e_i$, the exponents λ_i' and λ_i'' must be restricted in value. If neither a nor b is zero, both exponents must be positive integers (including zero) or each must be the half of an odd positive integer. If, on the other hand, either a or b is zero, y_1 is one of the fundamental integrals for e_i, and only the single exponent which belongs to this integral is thus restricted. It is necessary, therefore, that at least one exponent of each singular point in the finite plane, say λ_i'', shall be equal to the half of a non-negative integer. Also, since the square of y_1 is a polynomial of the n^{th} degree, one of the two exponents for infinity, say λ_∞'', must be equal to $-\frac{n}{2}$. The proposed solution can therefore now be expressed in the form $\Pi (x - e_i)^{\lambda_i''} Y$, where Y is a polynomial whose degree is $n' = \frac{n}{2} - \Sigma\lambda_i'' = -(\lambda_\infty'' + \Sigma\lambda_i'')$. The substitution of this in [1] gives as the differential equation for Y

$$\frac{d^2 Y}{dx^2} + \Sigma \frac{1 - \lambda_i}{x - e_i} \frac{dY}{dx} + \frac{(\lambda_\infty' + \Sigma\lambda_i'')(\lambda_\infty'' + \Sigma\lambda_i'')\, x^{r-2} + A'x^{r-3} + B'x^{r-3} + \ldots}{\Pi (x - e_i)} Y = 0, \qquad [7]$$

where λ_i is the exponent-difference $\lambda_i' - \lambda_i''$. This equation has been shown by Heine* to admit of a polynomial solution of degree n', provided the parameters A', B', ... are properly determined, and the number of such determinations for any given set of exponent-differences λ_i is

$$(n', r-1) = \frac{(n'+1)(n'+2)\ \dots\ (n'+r-2)}{1.\quad 2.\quad \dots\quad (r-2)}.\dagger \qquad [8]$$

We conclude therefore that *the differential equation* [1] *will admit of a particular solution whose square is a polynomial of the* n^{th} *degree only when the exponents satisfy the following conditions:*

(1). *One exponent* λ_i'' *of each singular point in the finite plane must be half of a non-negative integer.*

(2). $\frac{n}{2} - \Sigma\lambda_i''$ *must be a non-negative integer* n'.

(3). *One exponent of the singular point at infinity must be equal to* $-\frac{n}{2}$.

When any set of exponents is given which conform to these conditions, the number of such equations will be $(n', r-1)$.

It will be noticed that when neither a nor b is zero, the exponent-difference λ_i must be an integer. The logarithmic term, which ordinarily appears in the expansion of $P^{\lambda_i'}$ or $P^{\lambda_i''}$ when this is the case, must necessarily be eliminated by the conditions imposed upon the accessory parameters; that is, e_i is an *apparent singular point*. Furthermore, since neither exponent is negative, it follows that e_i cannot be an infinity of any solution of [1]. Hence the product of any two solutions will be holomorphic in the vicinity of the point.

§3. The simplest application of this result is to the differential equation for the hypergeometric series $F(\alpha, \beta, \gamma, x)$. The exponents for this equation are $\begin{pmatrix} 0 & \infty & 1 \\ 1-\gamma & \alpha & \gamma-\alpha-\beta \\ 0 & \beta & 0 \end{pmatrix}$. If, therefore, n is even, the sufficient condition is that α or β shall be equal to $-\frac{n}{2}$; if n is odd, not only must α or β be equal to $-\frac{n}{2}$, but either $1-\gamma$ or $\gamma-\alpha-\beta$ must be the half of an odd positive integer not greater than $\frac{n}{2}$. These results comprise four of the six cases given by Markoff in which the product of two solutions of the equation is a polynomial of the n^{th} degree. In two of these four cases he fails, however, to notice that the polynomial is the square of a single solution.

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* Berliner Monatsberichte, 1864, or Handbuch der Kugelfunctionen, Bd. I, s. 478.

† If $r = 2$, this number is unity.

§4. We have now to consider the conditions under which the product of two distinct solutions will be a polynomial. Let the requirement be first made that it shall be finite and one-valued. In the vicinity of e_i it will have the form

$$y_1 y_2 = a\,(P^{\lambda_i'})^2 + b\,(P^{\lambda_i''})^2 + cP^{\lambda_i'}P^{\lambda_i''}.$$

If neither a nor b nor c is zero, it can be argued in the same manner as before, that the exponents are both non-negative integers or are each the half of an odd positive integer, and that e_i is again an apparent singular point, in the vicinity of which every product of two integrals is holomorphic. The same conclusion holds if either a or b singly is zero. If c is zero, the only condition is that the two exponents are each the half of a non-negative integer. Hence unless e_i is again an apparent singular point, one exponent must be half of an odd positive integer and the other a non-negative integer. Finally, if a and b are both zero, $\lambda_i' + \lambda_i''$ shall be a non-negative integer. *Setting aside the apparent singular points, we have then some such scheme as*

$$\left(\begin{array}{cccc} e_1 & e_2 & e_3 & \ldots\ldots e_r \\ \frac{1}{2} + m_1' & \frac{1}{2} + m_2' & \lambda_3' + \lambda_3'' = m_3 & \ldots\ldots \\ m_1'' & m_2'' & & \ldots\ldots \end{array}\right)$$

for the exponents of the singular points in the finite plane, the m being zero or positive integers.

Such a scheme suffices to ensure at each of the points separately the existence of a one-valued finite product which has either the form $a\,(P^{\lambda_i'})^2 + b\,(P^{\lambda_i''})^2$ or $cP^{\lambda_i'}P^{\lambda_i''}$. We have next to learn under what conditions the product of two integrals will be one-valued when x makes a circuit around two singular points. Let e_1 and e_2 be two singular points whose circles of convergence overlap, and suppose also their exponents to have the values written down in the above scheme. Place

$$\left.\begin{aligned} P^{\lambda_1'} &= \alpha P^{\lambda_2'} + \beta P^{\lambda_2''}, \\ P^{\lambda_1''} &= \gamma P^{\lambda_2'} + \delta P^{\lambda_2''}. \end{aligned}\right\} \qquad [9]$$

In the vicinity of e_1 the product can be expressed as

$$a_1 (P^{\lambda_1'})^2 + b_1 (P^{\lambda_1''})^2 + c_1 P^{\lambda_1'} P^{\lambda_1''},$$

in the vicinity of e_2 as

$$(a_1\alpha^2 + b_1\gamma^2 + c_1\alpha\gamma)(P^{\lambda_2'})^2 + (a_1\beta^2 + b_1\delta^2 + c_1\beta\delta)(P^{\lambda_2''})^2 \\ + (2a_1\alpha\beta + 2b_1\gamma\delta + c_1\alpha\delta + c_1\beta\gamma) P^{\lambda_2'} P^{\lambda_2''}. \quad [10]$$

By a circuit about e_1 the sign of c_1 is changed; by one about e_2, the sign of the coefficient of $P^{\lambda_2'} P^{\lambda_2''}$. Comparing [10] with its value after both changes have been made, we obtain as the conditions that the product shall remain unaltered by a circuit around the two points,

$$c_1 = 0, \quad a_1\alpha\beta + b_1\gamma\delta = 0. \quad [11]$$

There is, therefore, save for a numerical factor, one product of two integrals, and only one, which remains unaltered for a circuit about e_1 *and* e_2.* In the region common to the two circles of convergence this product can be written in either of the forms

$$a_1 (P^{\lambda_1'})^2 + b_1 (P^{\lambda_1''})^2, \quad (a_1\alpha^2 + b_1\gamma^2)(P^{\lambda_2'})^2 + (a_1\beta^2 + b_1\delta^2)(P^{\lambda_2''})^2,$$

which shows that *the product is also unaltered for a circuit around* e_1 *and* e_2 *separately.*

There remain yet two other possible exponent-schemes for e_1 and e_2 to be examined, namely, $\left(\begin{array}{cc} \frac{1}{2} + m_1' & \\ & \lambda_2' + \lambda_2'' = m_2 \\ m_1'' & \end{array}\right)$ and $(\lambda_1' + \lambda_1'' = m_1, \ \lambda_2' + \lambda_2'' = m_2)$, but in neither case can a one-valued product be obtained without a specialization of the accessory parameters of the differential equation. For, assuming the first case, $a_1 (P^{\lambda_1'})^2 + b_1 (P^{\lambda_1''})^2$ must in the vicinity of e_2 become equal to $c_2 P^{\lambda_2'} P^{\lambda_2''}$. But if $c_1 = 0$, the coefficients of $(P^{\lambda_2'})^2$ and $(P^{\lambda_2''})^2$ in [10] can vanish only when $\begin{vmatrix} \alpha^2 & \beta^2 \\ \gamma^2 & \delta^2 \end{vmatrix} = 0$, and this imposes a condition upon the parameters of the differential equation. On the second assumption $P^{\lambda_1'}$, $P^{\lambda_1''}$ in the vicinity of e_2 can differ from $P^{\lambda_2'}$, $P^{\lambda_2''}$ only by constant factors, and this involves a two-fold specialization of the parameters.

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* In case the two circles of convergence do not overlap, the reasoning still holds good. The right-hand members of [9] must then be taken to represent what the left-hand members become, when continued analytically along some definite path to the vicinity of e_1.

The conclusions which have been reached for the singular points in the finite plane apply with only slight modifications to the point ∞. When the product of two integrals is here one-valued, either (1) the point is an apparent singular point, and λ'_∞, λ''_∞ are congruent both to $\frac{1}{2}$ or both to 0, mod. 1; or (2) they are congruent to $\frac{1}{2}$ and 0 respectively; or (3) $\lambda'_\infty + \lambda''_\infty$ is an integer. The exponents must be still further restricted if the product is a polynomial of the n^{th} degree. When expanded in series for $x = \infty$, it begins with $\left(\frac{1}{x}\right)^{-n}$. Hence in the first two of the three cases just specified, the exponent which is the smaller algebraically must be $-\frac{n}{2}$, and in the third case the sum of the two exponents must be $-n$.

§5. These considerations suffice for the solution of our problem, when there are three singular points e_1, e_2, ∞. The differential equation then contains no accessory parameter. To obtain a one-valued product we are therefore limited to taking two pairs of exponents which differ by the half of an odd integer. To make this product a polynomial, the exponents must also be so chosen that the product shall be finite in e_1 and e_2 and have at ∞ a pole of the n^{th} order. Accordingly we can take for the exponents either of the two following sets of values, but no others:

$$\text{I}\quad\begin{pmatrix} e_1 & e_2 & \infty \\ \frac{1}{2} + m'_1 & \frac{1}{2} + m'_2 & \lambda'_\infty + \lambda''_\infty = -n \\ m''_1 & m''_2 & \end{pmatrix},$$

$$\text{II}\quad\begin{pmatrix} e_1 & e_2 & \infty \\ \frac{1}{2} + m'_1 & \lambda'_2 + \lambda''_2 = m_2 & \frac{n_\infty}{2} \\ m''_1 & & -\frac{n}{2} \end{pmatrix}, \quad n_\infty > -n$$

the m being positive integers and n_∞ an integer, positive or negative, so chosen as to make in agreement with [2] the sum of the six exponents equal to unity.

The first of these exponent schemes comprises those equations which can be reduced by elementary transformations to the hypergeometric form without destroying the polynomial form of the product. For if $m''_1 = m''_2 = 0$ and $e_1 = 1$, $e_2 = 0$, we have at once the hypergeometric equation. When these constants have other values, an entire linear transformation of the independent variable will reduce e_1, e_2 to 0, 1, and the substitution

$$y=(x-e_1)^{\bar{\lambda}_1}(x-e_2)^{\bar{\lambda}_2}\bar{y}, \qquad [12]$$

in which $\bar{\lambda}_1$, $\bar{\lambda}_2$ denote respectively the smaller of the two exponents at e_1, e_2 will reduce one of the exponents at each of these points to zero. Applying, in particular, the exponent scheme to the differential equation for $F(\alpha, \beta, \gamma, x)$, we see that the product of two distinct solutions of that equation will be a polynomial, when α, β, γ have values in accordance with the following scheme:

$$\left(\begin{array}{ccc} 0 & \infty & 1 \\ 1-\gamma=\frac{1}{2}+m_1 & \alpha+\beta=-n & \gamma-\alpha-\beta=\frac{1}{2}+m_2 \\ 0 & & 0 \end{array}\right), \quad m_1+m_2=n.$$

This scheme embraces the two cases distinguished in Markoff's investigation, which were not included under §3.

§6. The same line of reasoning may be applied to a differential equation

$$\frac{d^2y}{dx^2}+\left(\frac{1-\lambda_1'-\lambda_1''}{x-e_1}+\frac{1-\lambda_2'-\lambda_2''}{x-e_2}\right)\frac{dy}{dx} + \left(\frac{\lambda_1'\lambda_1''}{(x-e_1)^2}+\frac{\lambda_2'\lambda_2''}{(x-e_2)^2}+\frac{A+Bx+Cx^2+\ldots}{(x-e_1)(x-e_2)}\right)y=0$$

with two singular points in the finite plane and an essential singularity at ∞. The product of two solutions will be holomorphic when

$$\begin{array}{ll} \lambda_1'=\frac{1}{2}+m_1', & \lambda_2'=\frac{1}{2}+m_2', \\ \lambda_1''=m_1'', & \lambda_2''=m_2''. \end{array}$$

To this form both "the differential equation of the functions of the elliptic cylinder"

$$\frac{d^2y}{d\phi^2}=(A\cos^2\phi+B)\,y$$

and also the equation

$$\frac{d^2y}{d\phi^2}=(A+B\cos 2\phi+C\cos 4\phi+\ldots)\,y$$

which Hill uses in his calculation of the motion of the lunar perigee, "so far as it depends on the mean motions of the sun and moon," can be reduced by the substitution $x = \cos 2\phi$. The resulting finite singular points and exponents are

$$\begin{pmatrix} +1 & -1 \\ \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}.$$

§7. The case in which there are four singular points can be discharged with almost equal rapidity. The sum of the eight exponents is 2 and the differential equation contains one arbitrary parameter. Consider first the following scheme of exponents:

$$\text{III}\begin{pmatrix} e_1 & e_1 & e_3 & \infty \\ m'_1 + \frac{1}{2} & m'_2 + \frac{1}{2} & m'_3 + \frac{1}{2} & \frac{n_\infty}{2} \\ m''_1 & m''_2 & m''_3 & -\frac{n}{2} \end{pmatrix},$$

in which the m and n_∞ have the same significance as before. It has been previously demonstrated that, except for a multiplicative constant, there is one, and only one, product whose value is independent of a circuit about two singular points, and that the same product is independent of a circuit about either separately. Since a circuit about two points is at the same time a circuit around the other two, it follows that there is one, and only one, product which is one-valued over the entire plane. The exponents show that it is everywhere finite except at ∞, where it has a pole of order n. It is therefore a polynomial of the n^{th} degree. Special interest attaches to this case, since no restriction whatever has been placed upon the arbitrary parameters. We shall subsequently see that this is impossible when the number of singular points is greater than four.

The general differential equation given by the foregoing scheme includes Hermite's equation as a special case. To obtain the latter we have only to place $m'_1 = m''_1 = \ldots\ldots = m''_3 = 0$ and $n_\infty = n + 1$. As already noticed, the substitution $x = p(u)$ will remove the first derivative from this equation and reduce it to the form

$$\frac{d^2y}{du^2} = [n(n+1)p(u) + h]\,y.$$

A corresponding reduction can be made in the more general equation. First, by a substitution similar to [12], we may reduce the differential equation to one which has an exponent-scheme of the form III but in which one exponent m''_i

of each finite singular point is equal to zero. When this is done, the substitution of the new independent variable

$$u = \tfrac{1}{2}\int \frac{dx}{(x-e_1)^{-m_1'} \ldots . (x-e_3)^{-m_3'}\sqrt{(x-e_1)\ldots(x-e_3)}}, \qquad [13]$$

which makes x an elliptic function of u, say $p_1(u)$, will remove the second derivative and reduce the equation to the form

$$\frac{d^2y}{du^2} = \frac{n_\infty n p_1 + B}{(p_1-e_1)^{2m_1'} \ldots . (p_1-e_3)^{2m_3'}}.$$

§8. A second group of equations with four singular points can be obtained by combining with two such singular points as occur in III an apparent singular point. Since the sum of the eight exponents is 2, two exponents for the fourth singular point must be chosen whose sum is an integer. According as the apparent singular point is at ∞ or in the finite plane, the exponents will therefore be

$$\text{IV}\begin{pmatrix} m_1' + \frac{1}{2} & m_2' + \frac{1}{2} & & \frac{n_\infty}{2} \\ & & \lambda_3' + \lambda_3'' = m_3 & \\ m_1'' & m_2'' & & -\frac{n}{2} \end{pmatrix},$$

$$\text{V}\begin{pmatrix} m_1' + \frac{1}{2} & m_2' + \frac{1}{2} & \frac{m_3'}{2} & \\ & & & \lambda_\infty' + \lambda_\infty'' = -n \\ m_1'' & m_2'' & \frac{m_3''}{2} & \end{pmatrix}.$$

The accessory parameters in the differential equation will in either case be determined by the condition that the logarithmic term in the expansions for the apparent singular point must be made to vanish.

With the first of these two exponent schemes a differential equation first given by Brioschi* and later applied by Haentzschel† to the theory of potential is closely connected. Haentzschel's form of the equation is

$$\frac{d^2y}{du^2} = [(m^2 - \tfrac{1}{4})p(u) - h]\,y,$$

in which m is an integer equal to Brioschi's $\frac{n-1}{2}$. If we free the equation from doubly periodic coefficients by the substitution $x = p(u)$, it becomes

$$(4x^3 - g_2x - g_3)\frac{d^2y}{dx^2} + (6x^2 - \tfrac{1}{2}g_2)\frac{dy}{dx} - [(m^2 - \tfrac{1}{4})x - h]\,y = 0.$$

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*Annali di Matematica, Serie 2, t. 9.

† "Studien über die Reduction der Potentialgleichung," p. 54.

Both writers prove that this equation admits of two integrals whose product is a polynomial multiplied into $\sqrt{x-e_i}$. Brioschi, however, appears to leave h arbitrary, an oversight which is corrected by Haentzschel. The exponent scheme for the equation is $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}(m+\frac{1}{2}) \\ 0 & 0 & 0 & -\frac{1}{2}(m-\frac{1}{2}) \end{pmatrix}$, but by setting $y=\sqrt[4]{x-e_3}\,\bar{y}$ it may be reduced to $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{m+1}{2} \\ 0 & 0 & -\frac{1}{4} & -\frac{m-1}{2} \end{pmatrix}$ and thus brought under IV. Brioschi gives the equation as an instance in which the square of the product of two solutions is a polynomial, but the modification just made shows that the equation does not differ essentially from those which we are here considering. Indeed, more generally, whenever the product of two solutions of a regular differential equation containing any number of singular points is equal to a polynomial multiplied by a product of powers of the binomial $x-e_i$, these factors may be removed and the equation reduced to the form treated in this paper by an appropriate substitution of the form

$$y=\Pi\,(x-e_i)^{a_i}\,\bar{y}. \tag{14}$$

§9. In the general case, where r, the number of singular points in the finite plane, is greater than 3, the differential equation contains $r-2$ accessory parameters. On these we are at liberty to impose an equal number of conditions in order to secure, if possible, a polynomial product. The consistency of the conditions thus imposed will have its verification in the existence of the polygons hereafter to be introduced in connection with the conformal representation of η, the quotient of two solutions. Consider first the case in which the exponents are

$$\text{I}\quad \begin{pmatrix} \frac{1}{2}+m_1' & \frac{1}{2}+m_2' \cdots\cdots \frac{1}{2}+m_r' & \frac{n_\infty}{2} \\ m_1'' & m_2'' \quad \cdots\cdots \frac{1}{2}+m_r'' & -\frac{n}{2} \end{pmatrix}.$$

We have seen that, irrespective of the values of the accessory parameters, there is one product of two integrals which is one-valued for circuits around e_1 and e_2. Let it be required that this product shall be one-valued for circuits around the remaining $r-1$ singular points. If r is even, the exponent difference for the point ∞ is an integer. One condition must consequently be imposed to remove from $P^{\lambda_\infty'}$ or $P^{\lambda_\infty''}$ the logarithmic term which would naturally appear. This leaves $r-2$ singular points, all of the same character, and $r-3$ independent parameters. If r is odd, one exponent for ∞ is an integer and the other is half of an odd integer. Whether then r is even or odd, the singular points which remain for consideration are all of the same character, and their number exceeds by a unit the number of remaining parameters. Of these singular points, two may be disregarded, for it has been shown that when the sum of the exponents of each of the two points is the half of an odd integer, the product of two integrals will be one-valued for circuits around these points, provided it is one-valued for circuits around every other point. The number of singular points left is therefore now one less than the number of parameters. At each of these points let $y_1 y_2$ be expressed in the form $a_i(P^{\lambda_i'})^2 + b_i(P^{\lambda_i''})^2 + cP^{\lambda_i'}P^{\lambda_i''}$. The values of the coefficients here obviously depend upon the accessory parameters of the differential equation. The condition that the product shall be one-valued over the entire plane requires that each coefficient c_i shall vanish. Since this imposes a single condition upon the parameters for each remaining singular point, a one-valued product can be obtained by imposing a total number of conditions which is one less than the number of parameters. When this is effected, the values of the exponents ensure that the product will be a polynomial. *To each set of exponents* I *there belongs therefore a differential equation containing a single arbitrary parameter, for which the product of two particular solutions will be a polynomial.*

This result may be regarded as an extension of one obtained by Brioschi for differential equations in which the coefficient of $\frac{dy}{dx}$ is one-half of the derivative of the coefficient of $\frac{d^2y}{dx^2}$. Such an equation is evidently obtained by placing all the m of scheme I equal to 0.

Similar considerations apply to such exponent schemes as

$$\text{II}\ \begin{pmatrix} \frac{1}{2}+m_1' \ldots\ldots \frac{1}{2}+m_r' & \lambda_\infty'+\lambda_\infty''=-n \\ m_1'' \qquad\qquad m_r'' & \end{pmatrix},$$

$$\text{III}\ \begin{pmatrix} \frac{1}{2}+m_1' \ldots\ldots \frac{1}{2}+m_{r-1}' & & \frac{n_\infty}{2} \\ & \lambda_r'+\lambda_r''=m & \\ m_1'' \ \ldots\ldots\ \ m_{r-1}'' & & -\frac{n}{2} \end{pmatrix}.$$

Since, however, the introduction of a singular point whose exponent-sum is an integer imposes two conditions upon the parameters, the total number of conditions will be equal to the number of accessory parameters. They will, therefore, be completely determined. It follows also that a scheme with more than one pair of such exponents will be in general impossible. It is, however, conceivable that in exceptional cases the conditions imposed at the several singular points might not all be independent. Cases may therefore arise where more than one such pair of exponents is present, as will indeed be obvious later when the conformal representation is considered.

This exhausts the possibilities of our problem except in so far as apparent singular points are introduced instead of those whose exponent-sums are the halves of odd integers. This can be done, since an apparent singular point, like the point it replaces, imposes but a single condition upon the accessory parameters. The number of points whose exponent-sums are the halves of odd integers must not, however, be made less than 2.

II.

§10. To distinguish briefly between the singular points whose exponent-sums are the halves of odd integers and those whose exponent-sums are integers, we will hereafter refer to them respectively as *singular points of the first and second kinds*. When the two solutions are distinct, we can, by a suitable substitution of the form [14], reduce the exponents for a singular point of the first kind to $\frac{1}{2} + m_i$, 0 and those for a singular point of the second kind to $\pm \frac{\lambda_i}{2}$ without destroying the property that the product of the two solutions is a polynomial. In the same manner the exponents for an apparent singular point can be reduced to zero and a positive integer. It becomes then what has been termed a semi-singular point, in the vicinity of which all solutions can be expanded in an ordinary power series. For convenience we will henceforth assume that these reductions have been made for all the singular points. The only effect of the reductions upon the polynomial is to remove from it all the factors $x - e_i$.

§11. At any singular point of the first kind the two solutions can be expressed as follows:

$$y_1 = C(\sqrt{a_i}\,P_i^0 + \sqrt{-b_i}\,P^{m_i+\frac{1}{2}}), \quad y_2 = \frac{1}{C}(\sqrt{a_i}\,P_i^0 - \sqrt{-b_i}\,P^{m_i+\frac{1}{2}}).$$

When x describes a circuit around the point, these will be changed into

$$\bar{y}_1 = C^2 y_2, \quad \bar{y}_2 = \frac{1}{C^2} y_1.$$

The result of a circuit around two such points is therefore to multiply the one solution by a constant ρ, *the other by its reciprocal* $\frac{1}{\rho}$. If, now, only singular points of the first kind are present, a hyperelliptic integral similar to [13] may be introduced as the independent variable in place of x. Since the periods of u are due to circuits of x around pairs of singular points, the proposition last enunciated shows that *there are two solutions of the differential equation, each of which is multiplied only by a constant whenever a period is added to* u. This theorem is well known in the case of Hermite's equation, the two solutions being then ordinary doubly periodic functions of the second class. In Hill's equation* the multiplication results upon the addition of the period 2π to the argument ϕ.

When the circles of convergence of the two singular points overlap, a formula can be given for the computation of ρ. Suppose the two points to be e_1, e_2. By a circuit around these points $\sqrt{a_1}\, P_1^0 \pm \sqrt{-b_1}\, P^{m_1+\frac{1}{2}}$ will be replaced by $(\sqrt{a_1}\alpha \mp \sqrt{-b_1}\gamma)\, P_2^0 - (\sqrt{a_1}\beta \mp \sqrt{-b_1}\,\delta)\, P^{m_1+\frac{1}{2}}$, or, expressed in terms of P_1^0, $P^{m_1+\frac{1}{2}}$ with the help of equations [9] and [11], by

$$(\sqrt{a_1}\, P_1^0 \pm \sqrt{-b_1}\, P^{m_1+\frac{1}{2}})\left(\frac{\alpha\delta + \beta\gamma \mp 2\sqrt{\alpha\beta\gamma\delta}}{\alpha\delta - \beta\gamma}\right).$$

We have therefore the formula

$$\rho = \frac{\alpha\delta + \beta\gamma \mp 2\sqrt{\alpha\beta\gamma\delta}}{\alpha\delta - \beta\gamma}. \qquad [15]$$

A circuit around an apparent singular point is obviously without effect upon the two solutions. On the other hand, near a singular point of the second kind, each solution is, except for a constant factor, identical with one of the two fundamental integrals, and they will therefore be multiplied, the one by $e^{+2i\pi\lambda_i}$ and the other by $e^{-2i\pi\lambda_i}$, where x describes a circuit around the point. Combining these results with the preceding we obtain the following noteworthy proposition:

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*See either Hill's article in the 8th volume of the Acta Mathematica or one by Callandreau, Astronomische Nachrichten, No. 2547.

If the two solutions, whose product is the polynomial, are selected as the bases of the monodromic group of substitutions of the equation, this group will take the form

$$\text{I.}\quad \bar{y}_1 = \sigma y_2, \quad \bar{y}_2 = \frac{y_1}{\sigma},$$

or

$$\text{II.}\quad \bar{y}_1 = \rho y_1, \quad \bar{y}_2 = \frac{y_2}{\rho}.$$

§12. The essential character of the group of a linear equation of the second order is more commonly exhibited by means of the quotient $\eta = \frac{y_1}{y_2}$. For the equation under discussion *the substitutions of* η *have the form*

$$\text{I.}\quad \bar{\eta} = \frac{\sigma^2}{\eta}, \text{ or II.}\quad \bar{\eta} = \rho^2\eta. \qquad [16]$$

In the Autographie of Klein's lectures upon "Linear Differential Equations," 1894, p. 148, a list of 11 cases is given in which the substitutions are simpler than the general substitution $\bar{\eta} = \frac{\alpha\eta + \beta}{\gamma\eta + \delta}$. Most of the differential equations which correspond to these cases are well known, as for instance the equations belonging to the groups of the regular solids. The chief case which has not received a general investigation is that in which the group has the form to which we have just been led by the consideration of the polynomial product. Conversely, if for any regular differential equation *the group of* $\eta = \frac{y_1}{y_2}$ *can be expressed in the form* [16], *the product of the two solutions* y_1, y_2 *must either be a polynomial or a polynomial multiplied by powers of the binomials* $x - e_i$, *and the latter case can evidently be reduced to the former by such a transformation as* [14]. The form of the substitutions of the group shows, in fact, that the product is multiplied by a constant when x describes a loop enclosing one or more singular points. Such a product is expressible as a holomorphic function multiplied into powers of the $x - e_i$ which correspond to the multiplicative constants and to the infinities of the product. Moreover, since the differential equation is supposed regular, the holomorphic factor must have a pole for $x = \infty$, and hence it is a polynomial.

§13. We shall hereafter confine our attention to *real* differential equations, i. e. to those in which all parameters, whether singular points, exponents, or accessory parameters, are real. Subscripts will be assigned to singular points of the first and second kinds according to the order in which they occur on the x-axis, the apparent singular points being, for convenience, omitted. We will now consider some properties of the solutions which relate to the segments into which the axis is thereby divided.

Consider first the four fundamental integrals which belong to the two extremities of any segment. Each integral has been defined by a power-series [3] which holds throughout a portion or the whole of this segment. Since the differential equation is real, the coefficients of each series must be real, and the signs in [3], which till now have been left arbitrary, can be so chosen that the integrals shall be real as long as the series converge. But any solution of the differential equation which is real along a finite portion of the axis, will, if continued analytically, remain real, until the first singular point is reached where an ordinary power-series fails to hold. The four fundamental integrals, when thus continued, will therefore be real throughout the entire segment irrespective of the apparent singular points which it contains. If, now, in [9], whether the circles of convergence of e_1 and e_2 overlap or not, the right-hand members of the equation are taken to represent what the left-hand members become when continued analytically from the vicinity of e_1 to that of e_2, the constants α, β, γ and δ must be real. It follows that ρ in formula [15] is either a real quantity or a complex imaginary with unit modulus according as $\alpha\beta\gamma\delta$ is positive or negative. *The substitutions which result from a circuit around two consecutive singular points e_{i-1} and e_i of the first kind must therefore be either both hyperbolic or both elliptic.*

Following a precedent set by Klein, we shall apply the terms hyperbolic and elliptic not only to the substitution but to the segment $e_{i-1}e_i$ around which the corresponding circuit is made. Equation [11] shows that the sign of $\alpha\beta\gamma\delta$ will be opposite to that of $\frac{a_1}{b_1}$. Hence in a hyperbolic segment the product can be expressed as $A_i^2(P_i^0)^2 - B_i^2(P^{m_i+i})^2$, and in an elliptic segment as $A_i^2(P_i^0)^2 + B_i^2(P^{m_i+i})^2$, in both of which A_i and B_i denote real constants. The two component solutions may therefore be so taken as to be real throughout a hyperbolic segment; on the other hand, in an elliptic segment, they will be conjugate imaginaries. A segment, one or both of whose extremities are singular points of the second kind, will here be classed with the hyperbolic segments, since in this segment both solutions can be taken as real. This follows from the fact that in the vicinity of such a point the two solutions differ only by constant factors from the two real fundamental integrals. On the other hand, the segments which terminate in a singular point of the first kind are the one elliptic and the other hyperbolic, because the sign of the second term of $A_i^2(P_i^0)^2 \pm B_i^2(P^{m_i+1})^2$ will be changed when x describes a circuit around e_i. The order of succession of the segments *between any singular point of the second kind and the next point of the same kind* is therefore a definite one. *The hyperbolic and elliptic segments alternate with each other, beginning and ending with a hyperbolic segment.* In agreement with this, the number of singular points of the first kind included between two consecutive points of the second kind must be even, as must also be the total number of points of the first kind.

A difference between the two varieties of segments again appears, when the roots of the polynomial are considered. In an elliptic segment the polynomial consists of the sum of two positive terms. Both of these cannot simultaneously vanish at any point of the segment, for if this were possible, two independent solutions of the differential equation would have at this point a common real root, which contradicts a well-known theorem concerning the alternation of the real roots. It follows therefore that *the real roots of the polynomial are situated only in the hyperbolic segments.*

§14. The foregoing theory can be advantageously set forth, and might, indeed, be independently developed, with the aid of the theory of conformal representation. As is well-known, the quotient η of two independent solutions of [1] builds the positive half of the x-plane conformally upon a polygon $E_1 E_2 \dots E_r E_\infty$ whose sides are arcs of circles. The angles at the vertices which correspond to the singular points e are successively equal to $\lambda_1 \pi, \dots, \lambda_\infty \pi$. The conformity of the representation ceases not only at the vertices but also at the points T of the boundary which correspond to the apparent singular points. The latter points will, however, not be here classed with the vertices of the polygon. The angle between the two arcs which meet in such a point is a multiple of π, and, because there is no logarithmic term in the expansion of η at an apparent singular point, the two arcs must be arcs of a common circle. Hence the point is to be regarded as a sort of turning-point (see Fig. 1) where the direction of a side is reversed one or more times.*

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* For a further discussion of such points, see my article in the 16th volume of the American Journal.

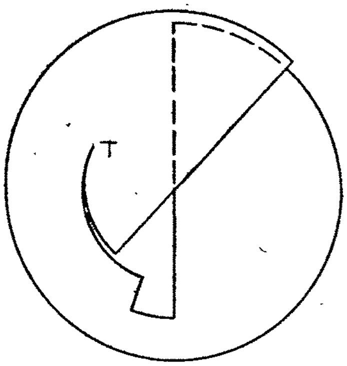
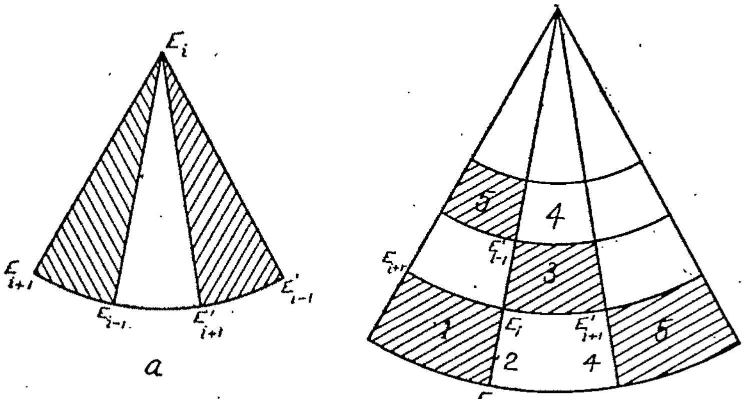


FIG. 1

The general shape of the polygon can be determined from the following considerations connected with the substitution-group of η. If the polygon be reflected on any one of its sides, we shall have a new polygon which is the image of the negative half-plane. A reflection of the second polygon upon one of its sides gives a second image of the positive half-plane which is connected with the first by a substitution of the group of η. If we suppose that the first reflection is on the side $E_{i-1}E_i$ and the second upon $E_i E'_{i+1}$ (Fig. 2), the substitution will be due to a circuit around e_i. The invariant points of this substitution will be the intersections of these two sides, produced if necessary, and hence also of the sides $E_{i-1}E_i$ and $E_i E_{i+1}$ of the first polygon. If e_i is a singular point of the second kind, the substitution is of the form (16, II), whose invariant points are $\eta = 0$ and $\eta = \infty$. The two sides $E_{i-1}E_i$ and $E_i E_{i+1}$ are therefore parts of straight lines which meet at the origin [Fig. 2(a)]. If the singular point is of



a

FIG. 2

b

the first kind, the substitution is of the form (16, I), whose invariant points, $\pm \sigma$, are symmetrically situated with respect to the origin. Since the angle at E_i is $(m_i+\frac{1}{2})\pi$, the two circles of which $E_{i-1}E_i$ and E_iE_{i+1} are arcs which cut each other in these points at right angles. But one of the singular points adjacent to e_i, say e_{i+1}, must likewise be a singular point of the first kind. It follows also that E_iE_{i+1} (see (b) of Fig. 2) must cut a second circle at right angles and in two points which are symmetrically situated with respect to the origin. Evidently therefore E_iE_{i+1} is the arc of a circle whose center is at the origin and $E_{i-1}E_i$ the segment of a straight line which passes through the origin, or *vice versa.* These conclusions concerning the shape of the polygon can be summed up in the following statement:

When the two solutions y_1, y_2 are distinct, the sides of the polygon are arcs of concentric circles and segment of straight lines which cut the circles at right angles.

§15. The methods by which polygons of this character are constructed will be discussed in a later paragraph. In the meantime some of the conclusions already obtained may be easily verified by means of the conformal representation. To a circuit around two consecutive singular points of the first kind corresponds a series of four reflections, as indicated in Fig. 2 (b). These result either in a simple revolution of the initial polygon through an angle ϕ or in increasing the distance of all its points from the origin in the ratio $\rho^2:1$. In other words, the resulting substitution is either elliptic or hyperbolic. Clearly also the straight sides correspond to the hyperbolic and the circular sides to the elliptic segments. The theorem which has been already given concerning the alternation of these two kinds of segments is now immediately evident from an inspection of the figures. Furthermore, the roots of y_1 and y_2 are respectively the zeros and the infinities of their quotient η. Hence we conclude *that if a side E_iE_{i+1} of the polygon passes p times in all through the zero and infinity points of the η-plane, the polynomial has p real roots situated between e_i and e_{i+1}; if also the interior of the polygon includes the zero and infinity points q times in all, q pairs of roots of the polynomial are imaginary.* Since also only the straight sides of the polygon can pass through the origin or infinity, the real roots must lie exclusively in the hyperbolic segments.

§16. The conformal representation also makes apparent the significance of the singular points of the second kind. Should one of the circular sides of a polygon be contracted to a point situated either at the origin or at ∞ (Fig. 3),

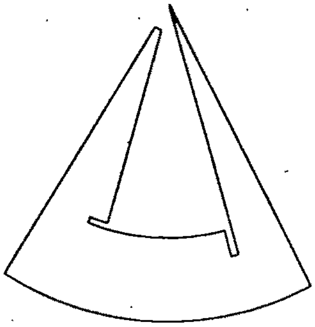


FIG. 3

the union of its extremities would evidently produce a vertex which would correspond to a singular point of the second kind. It is also obvious, conversely, that any such vertex can be regarded as having been formed in this manner. Hence *any differential equation with singular points of the second kind which satisfies the conditions of our problem can be regarded as the limit of an equation containing only singular points of the first kind, each singular point of the second kind being created by the union of two points which terminate an elliptic segment.* Thus, for example, when $m_1 = m_2 = 0$, the hypergeometric equations discussed at the close of §5 are limiting cases of Hermite's equation. It is sometimes possible, also without changing the angles of the polygon, to contract a circular side to a point which does not coincide either with the origin or infinity (see again Fig. 3). In such instances the contraction of an elliptic segment gives rise to an apparent singular point. The result is also the same when it is possible to shrink a hyperbolic segment to a point. From these instances it is clear that the various limiting forms of a given differential equation can be immediately inferred, when the shape of the corresponding polygon is known. In this respect, as in many others, the method of conformal representation has a decided superiority to analytical methods.

III.

§17. Our attention will now be restricted exclusively to such of our differential equations as contain only singular points of the first kind. If the number of these points is greater than 3, the differential equation will contain an arbitrary parameter which can be continuously varied. The polygon undergoes in consequence a continuous deformation, and the properties of the polynomial product also change. The present section will be devoted to a study of some of the changes in its properties which can be discovered by means of the conformal representation. Special attention will be paid to the changes in the distribution of the real roots of the polynomial among the segments of the axis of x.

§18. The general theory of these equations is similar to the well-known theory of Hermite's equation. When the parameter of the latter is continuously varied from $-\infty$ to $+\infty$, for certain critical values the two solutions forming the polynomial product become identical. The equation then becomes a Lamè's equation, and the two identical solutions, when divested of all factors $(x-e_1)^{\lambda_1}$, $(x-e_2)^{\lambda_2}$, $(x-e_3)^{\lambda_3}$, are simply Lamè polynomials. At the same time a change takes place in the distribution of the roots of the polynomial product among the segments of the axis. We will now show that for our more general differential equations the changes in the distribution of the roots occur only when the two solutions become identical. Since the coefficients of the polynomial are real, a change can be supposed to take place in only two ways: either (1) by the passage of one or more roots through a singular point from one segment into the next, or (2) by the conversion of pairs of real roots into conjugate imaginary roots. In the latter case a multiple real root must first be formed. But it is well-known that no solution can have a multiple root at a non-singular point of the plane, neither can two independent solutions have a common real root at such a point. It remains therefore only to examine when the polynomial has a root which coincides with a singular point. This again is impossible when the two solutions are distinct, because then in the vicinity of the point the polynomial may be written in the form $A^2(P_i^0)^2 \pm B^2(P_i^{m_i+1})^2$, only the second term of which vanishes for $x=e_i$. *The changes in the distribution of the roots of the polynomial can therefore take place only when the two solutions become identical.*

§19. When this is the case, a change simultaneously occurs in the character of the conformal representation. To determine the shape of the polygon we must take as before the quotient of two independent solutions. One of these, y_1, may be assumed to be, as in section I, the square root of the polynomial, and can accordingly be written in the form $(x - e_1)^{\varepsilon_1\lambda_1} \cdots\cdots (x - e_r)^{\varepsilon_r\lambda_r} P$, where each ε is either zero or unity and P denotes a polynomial which does not vanish at any singular point. Formula [5] then shows that the generating substitutions of the group of η will have the form

$$\bar{\eta} = \varepsilon^{2i\pi\lambda_i}\eta + \beta = -\eta + \beta .$$

One of the two invariant points of every such substitution is ∞, and it follows that every side of the polygon, produced if necessary, must pass through this point. *When, therefore, the two solutions forming the polynomial are coincident, the polygon is rectilinear.*

The position of the roots of the polynomial product can be directly deduced from the polygon. For it is clear from [5] that the roots of y_1 are the only infinities of η. *Hence if a side $E_i E_{i+1}$ of the polygon passes p-times through ∞, p roots are situated between e_i and e_{i+1}; if the interior of the polygon includes the point ∞ q times, q pairs of roots are imaginary; and lastly, if a vertex of the polygon is situated at ∞, the corresponding singular point is a root.* It will be noticed that each of these roots is a double root of the polynomial product unless it coincides with a singular point. In this case the order of its multiplicity is $2\lambda_i$.

§20. We have shown that in every instance the distribution of the roots among the segments is determined by the form of the polygon. To ascertain the changes in their distribution which result from a variation of the parameter, we have need therefore only to determine the changes in the shape of the polygon, and since a change can occur only when the two solutions become identical, it will suffice to follow the successive transitions through a rectilinear form. This will presently be done in detail for the case in which there are only four singular points.

§21. Before doing so, however, it is necessary to say a few words concerning the methods by which the polygons are constructed. The term polygon is to be understood in the broad sense in which it is employed in the Theory of Functions. As has been already said, the polygon may include either in its interior or on its boundary the point ∞. It will be necessary, therefore, in our diagrams to indicate upon which side of its boundary the polygon lies. This will be done by shading the diagrams. The polygon may also contain overlapping portions or leaves somewhat after the manner of a Riemann's surface. To facilitate the construction of the more complex polygons of this character, we shall have recourse to Klein's processes of attachment of circles or planes to polygons of simpler type. A polygon is said to be "*reduced*" when it cannot be constructed by such attachment from any simpler polygon. The different modes of attachment may be most easily illustrated by reference to Fig. 4, which represents the

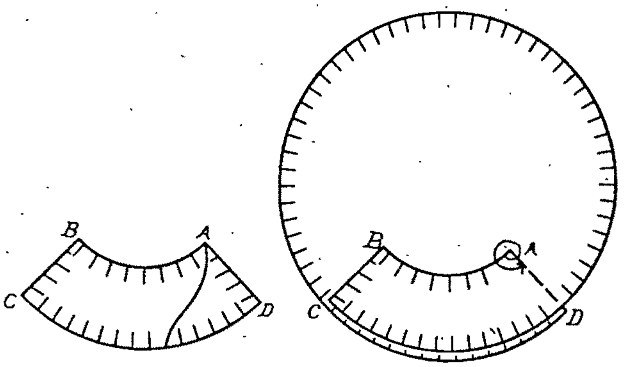


Fig. 4

simplest type of a reduced polygon of four sides. To increase A by 2π a circle is taken with the same radius as one of the opposite sides, say CD, and is placed above (or beneath) the polygon so that its boundary shall fall upon this side. The circle and polygon are then cut along a common line from A to CD, and the two are united across the cut like the two leaves of a Riemann's surface, the portion of either of which lies on one side of the cut being connected with the opposite portion of the other. In the resulting polygon the side CD must overlap itself. This process is known as the *polar attachment* of a circle, and may be repeated any number of times. If the same process be applied to increase the angle C, which, with the surface of the polygon, lies upon the convex side of AB, the portion of a plane exterior to a circle having the same radius as AB is to be employed. To cover such cases, the term circle, as in the Theory of Functions, will here be used to denote alike the portion of a plane within or without the bounding circumference. To increase two angles each by 2π, the process of *diagonal attachment* may be used. An entire plane is placed upon the polygon,

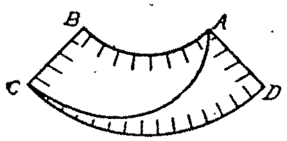


FIG. 5

the two are then cut along a common line between the vertices of the two angles (Fig. 5), and finally are connected in the manner before described. A third process, known as *lateral attachment,* increases each of two adjacent angles by π. Along the intervening side a circle is placed which has the same radius and which continues the surface of the polygon across this side. The connecting side is then erased so that the two figures form a continuous surface. Fig. 6 gives the result of such attachments on the sides CD and BC of Fig. 4. Two successive attachments on the same side are together equivalent to a single diagonal attachment of an entire plane between the two extremities of the side. It is to be observed that this attachment is not applicable to a side which overlaps itself. A fourth process, known as *transversal attachment,* adds to the polygon a circular ring, and is sufficiently explained by Fig. 7. The attachment can only be made to two sides,

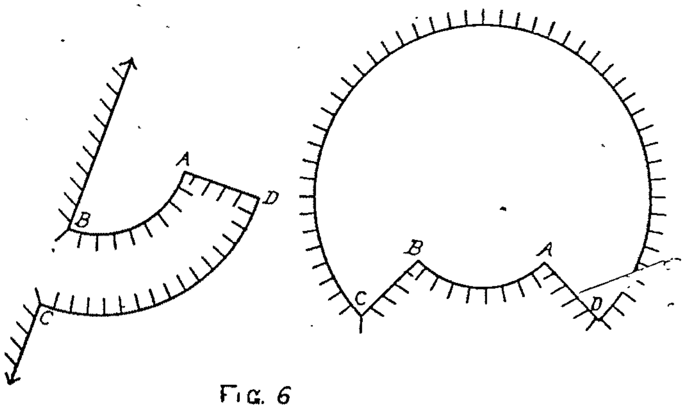


FIG. 6

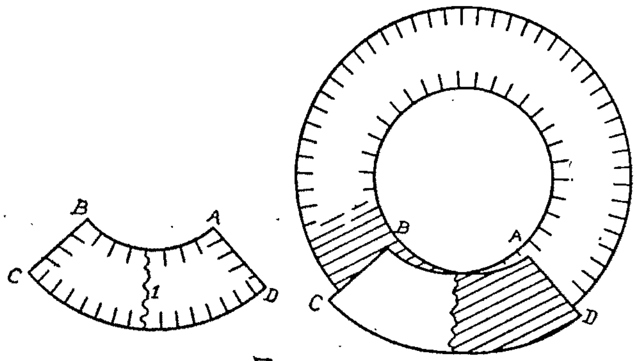


FIG. 7

which are arcs of non-intersecting circles, and leaves the angles of the polygon unaltered. The various attachments which have been described are not always possible, nor, when separately possible, are they always compatible one with another, but a glance at the polygon is usually sufficient to determine what system of attachments is applicable. It is therefore unnecessary to discuss the limitations upon their use further than to say that no cut can cross itself or any other cut. Whenever we have occasion to employ these attachments, they will be indicated merely by drawing the cuts and placing beside each cut a number to show how many attachments are to be made upon it. In the case of lateral attachment on any side, the number will be placed adjacent to the side.

§22. We may now return from our digression and take up the case of four singular points. The exponents in this case are

$$\begin{pmatrix} \tfrac{1}{2}+m_1 & \tfrac{1}{2}+m_2 & \tfrac{1}{2}+m_3 & \dfrac{n_\infty}{2} \\ 0 & 0 & 0 & -\dfrac{n}{2} \end{pmatrix}$$

and the differential equation takes the form

$$\frac{d^2y}{dx^2}+\left(\frac{\tfrac{1}{2}-m_1}{x-e_1}+\frac{\tfrac{1}{2}-m_2}{x-e_2}+\frac{\tfrac{1}{2}-m_3}{x-e_3}\right)\frac{dy}{dx}$$

$$+\left(\frac{-n_\infty nx+h}{4(x-e_1)(x-e_2)(x-e_3)}\right)y=0. \qquad [17]$$

If m_4 is used to designate the integral component of $\lambda_\infty=\frac{n_\infty+n_1}{2}$, it is easy to prove that *the sum of the four m is equal to the degree n of the polynomial.*

The polygon corresponding to this equation, whether it consist of one or many leaves, is in general a curvilinear quadrilateral bounded by two arcs of concentric circles and by two straight lines which cut the circles at right angles. By geometrical considerations, which will here be only briefly outlined, it can be shown that there are eleven types of reduced polygons of this character and no more. These are shown in Plate I. The apparent form of some of these types can be altered by the substitution $\bar{\eta}=\frac{1}{\eta}$, which exchanges y_1 and y_2, but for our purpose the original and the transformed polygon are obviously equivalent. In the case of types 2, 4 and 6, both forms of the polygon are presented.

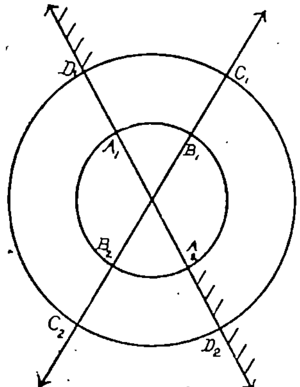


FIG. 8

The construction of these eleven types is based upon Fig. 8, which consists simply of two concentric circles cut by two straight lines through their common center. The vertices of the polygon must be selected, one from each of the four pairs of intersections A_1, A_2; B_1, B_2; C_1, C_2; D_1, D_2. We will first suppose that no side of the reduced polygon overlaps itself. If neither of the rectilinear sides passes through 0 or ∞, the one must be either B_1C_1 or B_2C_2 and the other D_1A_1 or D_2A_2. The boundary of the polygon has therefore the form represented in type 1. To show that the polygon itself must lie with reference to the boundary as represented in our diagram, it suffices to observe that if it were on the other side of the boundary it would contain the whole of the circle of which AB is an arc, and would therefore be reducible by a lateral detachment of this circle. These considerations, however, as yet only determine the angles to within multiples of 2π. But any other polygon, bounded in the same manner as the first polygon of the plate, would contain at least two angles which would exceed the corresponding angles of the latter by multiples of 2π, and would therefore permit of the diagonal detachment of one or more planes. Type 1 therefore represents the only type of reduced polygon which has no side which overlaps itself or passes through 0 or ∞. In the discussion of subsequent types similar reasoning will show, after the boundary of the polygon has been determined and also the side of the boundary upon which the polygon lies, that there is only one reduced polygon which meets the requirements. This will hereafter be assumed without further remark.

We proceed next to determine the reduced polygons which have but a single side which passes through either the origin or infinity. It is immaterial through which point the side is assumed to pass, since the points may be exchanged by the substitution $\bar{\eta} = \frac{1}{\eta}$. This side may therefore be taken as $D_1 \infty A_2$ or $D_1 \infty A_1$, and we may also suppose that the adjacent surface of the polygon is the border shaded in Fig. 8. The second rectilinear side must be either B_1C_1 or B_2C_2. If, now, the first side terminates in A_2, this vertex must be connected with B_1 and B_2 respectively by the arcs A_2B_1 and $A_2B_1B_2$, because otherwise the polygon would contain the whole of the circle lying within the circumference $A_2B_2B_1$ and would consequently be reducible. Completing, finally, the polygon by the addition of a fourth side, we obtain types 2, 3 and 4. The polygon in which the fourth side is the arc $C_2C_1D_1$ is excluded, because it would necessitate a winding point at D_1 and would therefore be reducible by lateral detachment along this side. If, on the other hand, the first rectilinear side terminates in A_1, the second cannot be B_2C_2. For if it were, the whole of the half-plane adjacent to the former side would be contained in the polygon and could be detached laterally. We have therefore only to connect $D_1 \infty A_1$ with B_1C_1, and this can be done in two ways, as shown in types 5 and 6.

If both the rectilinear sides of the polygon pass through the origin or infinity, we may distinguish the following cases:

(1). One rectilinear side $D_1 \infty A_2$ passes through ∞ and the other, $B_1B_2C_2$ or $B_2B_1C_1$, through the origin (Types 7 (a) and 7 (b)).

(2). Both sides pass through the origin or through ∞, say the origin. We have then to connect two such segments as $D_1A_1A_2$ and $B_1B_2C_2$ (Type 8).

(3). One side $D_1 \infty A_1$ passes through the origin and infinity, and the other only through the origin. The segment $B_1B_2C_2$ must be selected as the second side, since otherwise the polygon could be reduced by the lateral detachment of the half-plane adjacent to the former side (Type 9).

(4). Each side passes through the origin and infinity. With $D_1 \infty A_1$ must be associated the segment $B_1 \infty C_2$, since otherwise a half-plane could be removed (Type 10).

It remains now to consider the possible forms of a reduced polygon, one or more of whose sides overlap. Examples of such polygons can be obtained from two of the preceding types, namely, Types 4 and 7 (b) by prolonging the opposite arcs each by a semi-circumference. We shall, however, still consider the polygons to be of the same type. The only other types in which the arcs can be produced till they overlap are the 1st and 3d, but these polygons will then be reducible either by transversal or by polar detachment. There are therefore no other reduced polygons in which the sides overlap because the polygon winds in ring-form between the concentric arcs. We have therefore only to consider the cases in which the overlapping is effected in some other way. Since the surface overlaps at the same time as the side, it must wind around one or both of the vertices opposite to the side. If the side be rectilinear, it is easy to see (compare Fig. 9)

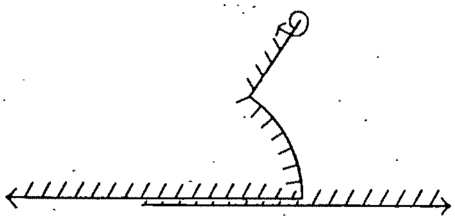


FIG. 9

that the surface makes a complete winding around one of the two vertices. The polygon can therefore be reduced by polar detachment. The same is true if a complete winding takes place around a vertex opposite to a circular side. There remains therefore only the case where there is a partial winding around both vertices, so that the angles here are $\frac{3}{2}\pi$. The only reduced polygon of this character which can be constructed is presented in Type 11. Our list of reduced polygons is therefore now complete.

Each reduced polygon gives rise by attachment to a system of polygons. Many of the polygons thus constructed can, however, be constructed from two or more distinct types. We will, for example, obtain the same form of polygon by a lateral attachment on BC in Type 3 as by a diagonal attachment between B and D in Type 4; or again, by a lateral attachment on DC in Type 8 as by a lateral attachment on BC in Type 7 (a).

§23. We are now prepared to construct for any given values of the m a polygon which corresponds to the differential equation [17], and to trace the successive changes in form, when the parameter h of the equation is continuously varied. A complete determination of the polygon depends, of course, upon the anharmonic ratio of the singular points as well as upon the accessory parameter. A two-fold variation in the form of the polygon is accordingly possible. Either the ratio of the radii of the two concentric arcs or the inclination of the two rectilinear sides may be continuously altered. We shall, however, take account only of such changes of form as affect the type of the reduced polygon and the corresponding system of attachments. With this understanding it will be first found that by continuous geometrical deformation a series of different forms is obtained, which succeed one another in definite order, and subsequently it will be shown that a variation of the parameter h alone gives rise to the series thus obtained. We may start with any polygon having the angles $(m_i + \frac{1}{2})\pi$, for from it all other forms of polygons with the same angles will be subsequently obtained. We will first consider the case in which some one of the m, say m_4, is equal to or greater than the sum of all the others.

$$\text{I.} \quad m_4 \geqq m_1 + m_2 + m_3.$$

To bring the m to a form corresponding to the system of attachments to be employed in the construction of the polygon, we shall avail ourselves of one of the four following arithmetical reductions, in which s, t, x and z denote integers, positive or zero.

$$(1) \quad \begin{aligned} m_4 &= 2s + 2t + x + z \\ m_1 &= x \\ m_2 &= 2t \\ m_3 &= z. \end{aligned}$$

$$(2) \quad \begin{aligned} m_4 &= 2s + 2t + x + z + 1 \\ m_1 &= x \\ m_2 &= 2t + 1 \\ m_3 &= z. \end{aligned}$$

$$(3) \quad \begin{aligned} m_4 &= 2s + 2t + x + z + 1 \\ m_1 &= x \\ m_2 &= 2t \\ m_3 &= z. \end{aligned}$$

$$(4) \quad \begin{aligned} m_4 &= 2s + 2t + x + z + 2 \\ m_1 &= x \\ m_2 &= 2t + 1 \\ m_3 &= z. \end{aligned}$$

The first two reductions are to be employed when the polynomial is of even degree; the last two, when the polynomial is of odd degree. In all four cases the form of the reduction shows that after the selection of a suitable reduced polygon, a system of attachments may be employed consisting of t diagonal attachments between E_2 and E_4, x and z lateral attachments on the sides E_4E_1 and E_4E_3 respectively, and s polar attachments from E_4 to one of the two opposite sides, say E_2E_3. In the first case we must select the first type of reduced polygon, in the remaining three cases types 7, 4 and 3 respectively, A, B, C and D being taken in each case as the vertices E_∞, E_1, E_2, E_3.

All possible changes in the form of the polygon for case 1 are shown in Plates II and III. As before pointed out, the essential features of the polygon are modified only by transition through a rectilinear form. It suffices therefore to indicate in our figures these successive transitions. The rectilinear forms are marked in the plates with even numbers, the intermediate stages with odd numbers. The passage to a rectilinear form is effected, of course, by withdrawing the center of the concentric arcs to ∞. With the exceptions to be hereafter noted, the successive transitions can be effected only in the order in which they are given in the plates.

The two plates, taken together, are divided into four sections, each of which illustrates a cycle of changes which is to be repeated as many times as possible. In the first cycle a polar attachment is transferred from E_2E_3 to E_2E_1, as is seen by a comparison of the first and last figures of the cycle. The lateral and diagonal attachments remain, however, unaltered. The corresponding index numbers have been inserted only in the first and last polygons, it being understood that in each intermediate polygon there is an equal number of lateral, as of polar, attachments. The second figure may be obtained from the first by withdrawing the center of the circular sides to ∞. The only new form which is possible when it reappears in the finite plane is that represented in Fig. 3 (a) or a similar figure in which E_1E_2 is the inner and $E_\infty E_3$ the outer arc. These two figures are, however, equivalent by virtue of the substitution $\bar{\eta} = \frac{1}{\eta}$. Figure 3 (b) is of the same form as 3 (a), one of the s polar attachments being explicitly represented. If, now, in this figure the center of the concentric arcs is carried to the right along the side E_2E_3 to ∞—if carried in the opposite direction, we return to Fig. 2—the vertices E_3 and E_2 both pass to ∞, but E_1 must remain in the finite plane, since otherwise the polygon would degenerate into a triangle. We thus arrive at Fig. 4. The passage thence to Fig. 6 requires no comment. The seventh and eighth polygons have been omitted, inasmuch as they can readily be supplied by the reader, being similar in structure to the fifth and fourth polygons respectively, but with an interchange in the roles of E_1 and E_3. Omissions of like character will likewise be made in subsequent cycles. This cycle is to be repeated s times, that is, until all the polar attachments have been transferred to E_1E_2. It may then be applied once more, until the reduced polygon 3 (a) is reached, when it will be found impossible to proceed further. The polygon thus obtained is the initial figure of the second section.

The second cycle removes a diagonal attachment and replaces it by a pair of polar attachments to E_1E_2. At the same time the number of lateral attachments on $E_\infty E_3$ is diminished and the number on E_3E_2 increased, each by two. The successive changes require no particular comment, until we reach the two polygons 5. These (as later other pairs of polygons) are numbered alike to call attention to the fact that, although constructed from two different types of reduced polygon, they are identical in form. On leaving this figure, two alternative courses are open, either to proceed as in the plate to the ninth polygon or to pass from the one to the other by means of 5 (b) and 5 (c) (see adjoining Fig. 10) without the insertion of any rectilinear form. Either succession of changes is geometrically possible, and a decision between them cannot here be made. Presumably it is dependent upon the position of the singular points upon the axis. The cycle can be repeated until either all the diagonal attachments or all the lateral attachments on $E_\infty E_3$ have been removed. The former is the case when $\frac{z}{2} \geqq t$, that is, $m_3 \geqq m_2$; the latter, when $m_3 \leqq m_2$.

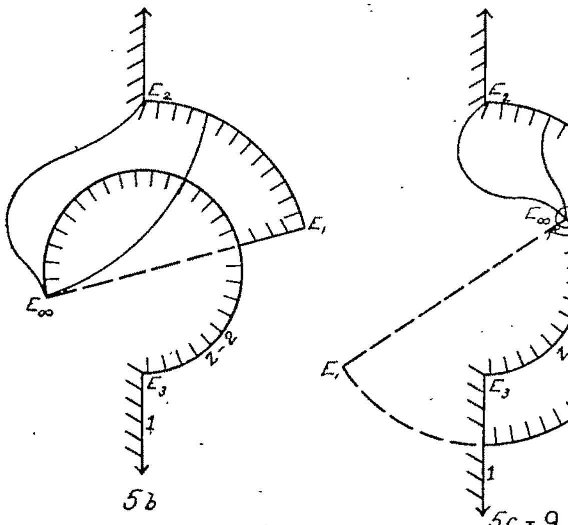


Fig. 10

The third cycle is applicable only when $m_3 \geqq m_2$, and the effect of its repeated application is to remove the remaining lateral attachments. The changes for the first half of the cycle are the same as in Figs. 1 to 5 of cycle 2. We then insert two new figures, numbered 5 (a) and 5 (b), and thence proceed as in polygons 10 to 13 of cycle 2 to the ninth and final figure of the cycle. Each half cycle removes a lateral attachment on $E_\infty E_3$ and replaces it, the one by a polar attachment from E_∞ to $E_1 E_2$, the other by a polar attachment from E_3 to the same side. According as the number of lateral attachments to be removed is odd or even, the reduced polygon with which we conclude the last application of this cycle will have the form given in 5 or in 9. Each of these polygons contains part of a circular ring included between the sides $E_\infty E_3$

and E_1E_2. Since all the lateral attachments have been removed from these sides, they can be indefinitely prolonged, thus adding an indefinite number of circular rings to the figure. This, as will be later shown analytically, is always the final outcome of an indefinite increase of the parameter.

When $m_3 \leqq m_2$, the prolongation of the two circular sides begins immediately upon conclusion of the second cycle, the reduced polygon being then either 5 or 13 of cycle 2. Owing, however, to the presence of diagonal attachments between E_∞ and E_2, this will not result at once in the addition of circular rings to the polygon. The last section of Plate III shows the effect of a prolongation of each of the circular sides for a complete circumference, a diagonal attachment being of necessity replaced by two polar attachments, the one from E_∞ to the side E_1E_2, the other from E_2 to the side $E_\infty E_3$. By a repetition of this process the diagonal attachments will be removed, but at any time before this has been accomplished another change in the form of the polygon may be made. By passage through a rectilinear form, the circular sides to which the polar attachments are made may be converted into the rectilinear sides. But if this is done, to continue the transformation of the polygon, it will be necessary to re-exchange the circular and rectilinear sides either by retracing our steps or by completing the series of changes as indicated in Fig. 11* of the text. The final outcome will be the same whether these changes be included or not. The diagonal attachments will eventually be all replaced by polar attachments, and the further prolongation of the circular sides $E_\infty E_3$ and E_1E_2 will thereafter result in the addition of circular rings ad infinitum.

We have now traced all possible changes in the form of the polygon upon the hypothesis that the parameter is varied continuously in one direction. It remains to consider what changes the polygon will undergo when the parameter is varied in the opposite direction. As already stated, the only difference between the first polygon of cycle 2 and the first of cycle 1 is that the polar attachments are all made to E_1E_2 in the one case and to E_2E_3 in the other. By a change of subscripts, the subsequent cycles will therefore apply equally well to either side of the first cycle. A variation of the parameter in the opposite direction will also ultimately result in the addition of circular rings, which, however, will be included between the sides $E_\infty E_1$ and E_2E_3.

---

* The transition from 3 (a) to 8 (b) is effected by increasing in the former polygon the radius of the inner arc E_3E_2 until it exceeds the radius of $E_\infty E_1$. The two rectilinear sides then overlap, as in polar attachment.

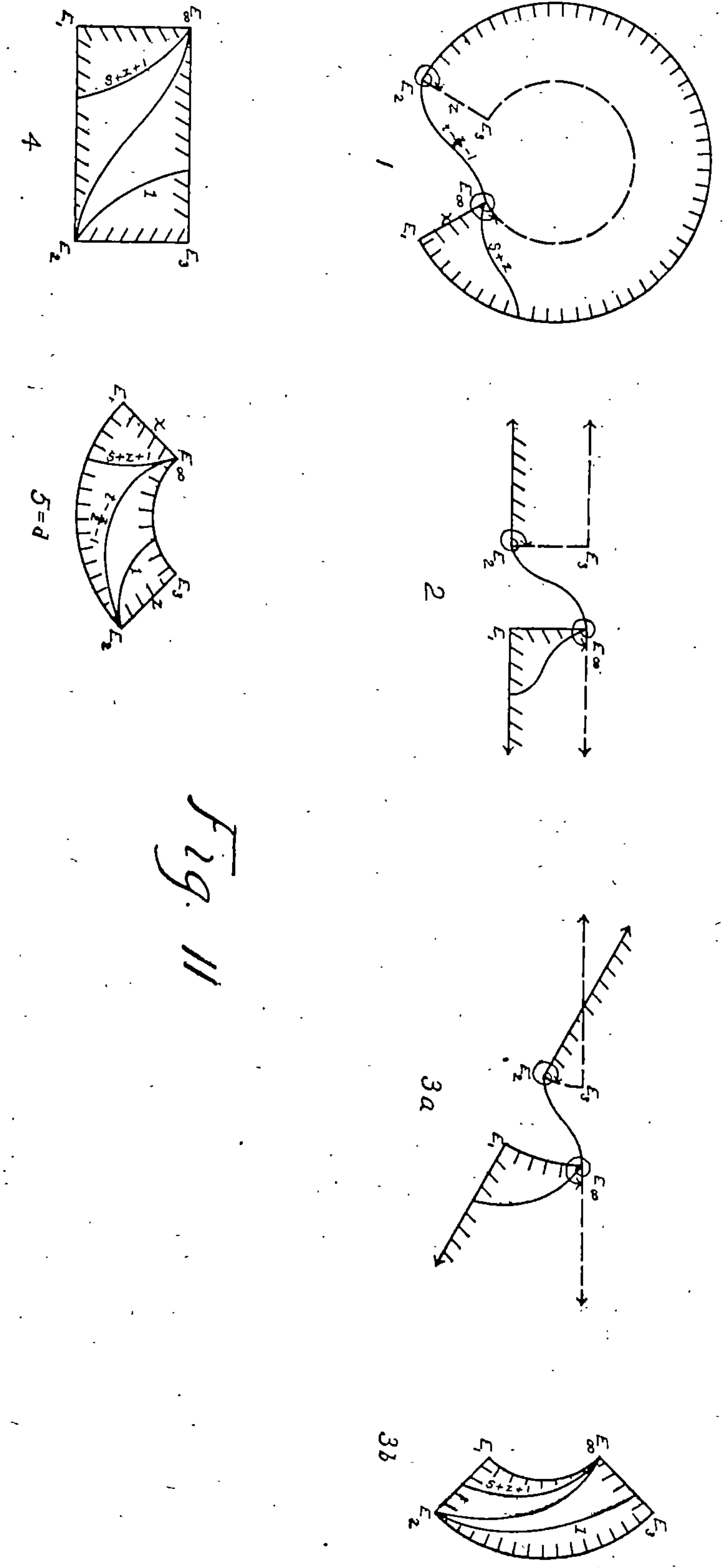


Fig. 11

This completes the discussion for case 1. The second case differs from the first in no essential feature. Plate IV gives the first cycle of changes. By its repeated application the polar attachments, as before, are transferred from E_2E_3 to E_1E_2. In the last application of the cycle it will be found that the seventh polygon is identical in structure with the fifth of cycle 2, case 1, except that there is no lateral attachment upon E_2E_3. From this point on, the discussion is the same as in case 1. In the subsequent figures there will be an even or an odd number of attachments on this side according as the number of such attachments was in the former case odd or even.

The changes for case 3 are shown in Plates V and VI and for case 4 in Plate VII. The successive cycles are in every way similar to those of the first two cases, and case 4* is related to case 3 precisely as case 2 to case 1.

§24. When no one of the m is greater than the sum of the remaining three, we may, without loss of generality, assume that

$$\text{II.}\quad m_4 + m_2 \geqq m_3 + m_1, \qquad m_4 + m_1 \geqq m_2 + m_3.$$

One of the four following arithmetical reductions may then be made, t, z and x being non-negative integers and y a positive integer.

$$\begin{array}{ll}
(5).\ m_4 = 2t + z + x & (6).\ m_4 = 2t + z + x + 1 \\
 m_1 = x + y & m_1 = x + y \\
 m_2 = 2t + y & m_2 = 2t + y + 1 \\
 m_3 = z. & m_3 = z. \\
 & \\
(7).\ m_4 = 2t + z + x + 1 & (8).\ m_4 = 2t + z + x + 2 \\
 m_1 = x + y & m_1 = x + y \\
 m_2 = 2t + y. & m_2 = 2t + y + 1 \\
 m_3 = z. & m_3 = z.
\end{array}$$

In the first two of these four cases the polynomial is of even degree; in the last two, of odd degree. The same reduced polygons may be selected as in the corresponding cases of I, namely, the first, seventh, fourth and third types, B, C, D and A being taken as before for the vertices E_1, E_2, E_3 and E_∞. The polygons are then completed by t diagonal attachments between E_∞ and E_2 and by x, y and z lateral attachments along the sides $E_\infty E_1$, E_1E_2 and $E_\infty E_3$. The polygon for case 8 can also be built up from Type 4, since the first lateral attachment on BC in Type 3 is equivalent to a diagonal attachment between B and D in Type 4, and the polygon will therefore differ essentially from that for case 7 only in the selection of the vertices E_i.

---

*The seventh polygon of cycle 1 of this case, after the polar attachments have all been transferred to E_1E_2, is converted into one similar to 5 (b) or 9 of cycle 2, case 3, by enlarging the radius of the inner arc until it exceeds that of the other arc.

The first cycle of changes for cases 5–7 is shown in Plate VIII. The changes in case 8 are similar to those in case 7. With each repetition of the cycle the number of lateral attachments on each of two opposite sides, E_1E_2 and E_3E_∞, is diminished by a unit, while the number on each of the other sides is increased a like amount. The cycle is to be repeated until all the lateral attachments have been removed from one of the first two sides. This side is then free for polar attachment, as was also E_2E_3 at the outset. The cycle is therefore to be both preceded and followed by other cycles in exactly the same manner as was cycle 1 in the corresponding cases of I. We may therefore limit our attention altogether to the present cycle of changes in the polygon, this being the only one of a new character.

§25. We have now seen for each case all possible changes in the form of the polygon. It remains to prove that when the parameter is continuously varied, the polygon will pass through the series of changes which have been described. For this it will evidently suffice to show that an indefinite increase or decrease of the parameter will result in the addition of an indefinite number of circular rings included in the one case between E_3E_∞ and E_1E_2, in the other, between $E_\infty E_1$ and E_2E_3. To demonstrate this we first reduce equation [17] by the substitution

$$y = (x - e_1)^{\frac{1}{2}m_1 - \frac{1}{2}} \dots . (x - e_3)^{\frac{1}{2}m_3 - \frac{1}{2}} \bar{y}$$

to the form

$$\frac{d^2y}{dx^2} + \left(R(x) + \frac{h}{4(x - e_1)(x - e_2)(x - e_3)}\right) y = 0, \qquad [18]$$

in which $R(x)$ is a rational fraction that is finite except at the singular points. If h is then taken sufficiently large, the coefficient of y will have for any given value of x the same sign as $\frac{h}{(x - e_1)(x - e_2)(x - e_3)}$. For large positive values of h the sign will therefore be positive in the segments e_1e_2 and $e_3\infty$, for large negative values in the segments ∞e_1 and e_2e_3. The coefficient can, moreover, be made greater than any given positive constant a. Now it is well known that every real solution of the equation $y'' + ay = 0$ has an infinite number of real roots which cumulate in both directions in the vicinity of the point at ∞. But, by a theorem of Sturm,* if G' and G'' are two functions of x which are finite and continuous for any interval of the axis of x, and if G' is algebraically less than G'', then between any two successive roots of a real solution of $y'' + G'y = 0$, which are situated in this interval, there must lie at least one root of every real solution of $y'' + G''y = 0$. It follows that *when h is indefinitely increased, any real solution of* [17] *will have an infinite number of roots in the segment $e_3\infty$, and when h is indefinitely decreased, an infinite number of roots in the segment ∞e_1.* Like results must also hold for the segments e_1e_2 and e_2e_3, respectively, since by a linear substitution the singular points e_2 and ∞ can be interchanged and at the same time the value of h is multiplied by a negative constant.† Thus, whether h is indefinitely increased or diminished, every real solution will have an indefinitely large number of roots in alternate segments of the axis. Furthermore these segments cannot be hyperbolic segments, because in such segments the two factors of our polynomial product are real solutions and its degree would then be infinite. In the elliptic segments the two solutions will have the form $AP_i^0 \pm \sqrt{-1}\, BP^{m_i+\frac{1}{2}}$, in which P_i^0 and $P^{m_i+\frac{1}{2}}$ will each have an infinite number of zeros, the zeros of P_i^0 alternating with those of $P^{m_i+\frac{1}{2}}$. Hence as x traverses either elliptic segment, the argument of η, which is equal to $2\tan^{-1}\dfrac{BP^{m_i+\frac{1}{2}}}{AP_i^0}\,\dfrac{B}{A}$, will increase without limit. It follows that when h approaches ∞, an indefinite number of complete circumferences will eventually be added to the circular sides of the polygon. Since the angles of the polygon remain unaltered, this can be done only by the successive addition of circular rings.

§26. Our figures may now be applied to a study of the polynomial product. First consider cases 1 and 2 in which the product is of even degree. Upon examination of the rectilinear polygons it will be found either that the vertices all lie in the finite plane or that two of the vertices E_1, E_2, E_3 are situated at ∞. For the critical values of the parameter the square root of the polynomial product can therefore be expressed in one of the four following forms:

---

* Lionville, tome I, p. 185.

† See my article in the Bulletin of the American Mathematical Society, June, 1898, p. 432.

$$
\begin{aligned}
&(1).\ P_{\frac{n}{2}},\\
&(2).\ (x-e_2)^{m_2+\frac{1}{2}}(x-e_3)^{m_3+\frac{1}{2}}P_{\frac{n}{2}-m_2-m_3-1},\\
&(3).\ (x-e_1)^{m_1+\frac{1}{2}}(x-e_3)^{m_3+\frac{1}{2}}P_{\frac{n}{2}-m_1-m_3-1},\\
&(4).\ (x-e_1)^{m_1+\frac{1}{2}}(x-e_2)^{m_2+\frac{1}{2}}P_{\frac{n}{2}-m_1-m_2-1}.
\end{aligned}
$$

The polynomials P thus introduced fall into four distinct classes, and those which belong to the same class are solutions of differential equations with common exponent-differences. According to Heine's formula [8] the number of polynomials in the several classes must be equal to

$$(1).\ \frac{n}{2}+1,\quad (2).\ \frac{n}{2}-m_2-m_3,\quad (3).\ \frac{n}{2}-m_1-m_3,\quad (4).\ \frac{n}{2}-m_1-m_2,$$

and the total number will be

$$2n+1-2\left[m_1+m_2+m_3\right]=4s+4t+2x+2z+\begin{cases}1, \text{ case } 1;\\ 3, \text{ case } 2.\end{cases}$$

The number of rectilinear polygons will not, however, necessarily be so great, inasmuch as they correspond only to real values of the accessory parameter, that is, to polynomials with real coefficients. The lower limit to the number of such polygons can be obtained by a count of the minimum number of rectilinear polygons included between the two polygons with series of ring attachments, and it will be found to be $4s+2x+2z\begin{cases}+1, \text{ case } 1\\ -1, \text{ case } 2\end{cases}$. *Our geometrical investigation furnishes*, therefore, for the cases under consideration, *a supplement to Heine's theorem.* The missing polynomials belong to the first and third classes.

An inspection of the plates also shows that the polynomials of the several classes recur in each cycle in a definite order. The first cycle is, however, the only one in which all four classes are included. The order in which they there recur is for case 1 the same as that in which they were above enumerated; in case 2 they recur in opposite order.

As before explained, the changes in the distribution of the real roots of the polynomial product which result from a continuous change of the parameter h can easily be traced by comparing successively each rectilinear polygon with the polygons which immediately precede and follow it. In the two cases before us, as also in all cases to be hereafter examined, each passage of the polygon through a rectilinear form exchanges the rectilinear with the circular sides. *As therefore the accessory parameter passes successively through the critical values, each segment of the axis will be alternately elliptic and hyperbolic.*

The successive changes in the position of the roots may be advantageously shown by a graphical representation such as was introduced by Klein in his discussion of Hermite's equation. For this purpose the values of h are plotted as ordinates and the roots of the corresponding polynomials as abscissas. The resulting curve $F(P_n, h) = 0$ shows at a glance the dependence of the roots upon the parameter h. Specimen sections of the curve, which correspond to the first applications of the various cycles, are given in the first half of Plate IX for case 1. Horizontal lines which represent the critical values of the parameter are added and numbered to correspond with the rectilinear polygons in Plates II and III. These, together with the vertical lines $x = e_1, e_2, e_3$, divide the plane into rectangles, in which alternately the two solutions are elliptic and hyperbolic To each successive repetition of cycle 1 corresponds a branch of the curve similar to that drawn in the plate, but the number of oscillations between e_2 and e_3 which corresponds to the number of polar attachments on the rectilinear side $E_2 E_3$ is every time diminished by a unit, and the number of oscillations between e_1 and e_2 is increased by a unit. In like manner the number of oscillations between e_1 and e_2 is increased by two units with each successive repetition of cycle 2 or 3. The dotted portions of the curve correspond to the series of polygons in cycles 2 and 4 whose presence cannot definitely be affirmed. The ovals are to be included in the curve only when the numbers which they enclose are odd. Below the first section of the plate are to be added sections similar in structure to those above it, but the roles of the segments $e_1 e_2$ and $e_2 e_3$ must be interchanged. The first section of the curve for case 2 is indicated at the foot of Plate IV. The second and third sections are similar to the corresponding sections of case 1 but with the insertion of an oval in the lowest rectangular space between $x = e_2$ and $x = e_3$.

The curve given by Klein for Hermite's equation is comprised under case 1, when the degree of the polynomial-product is even, and is the special case in which there is but a single cycle of changes. The curve therefore consists entirely of sections similar to that given for the first cycle, but without the ovals.

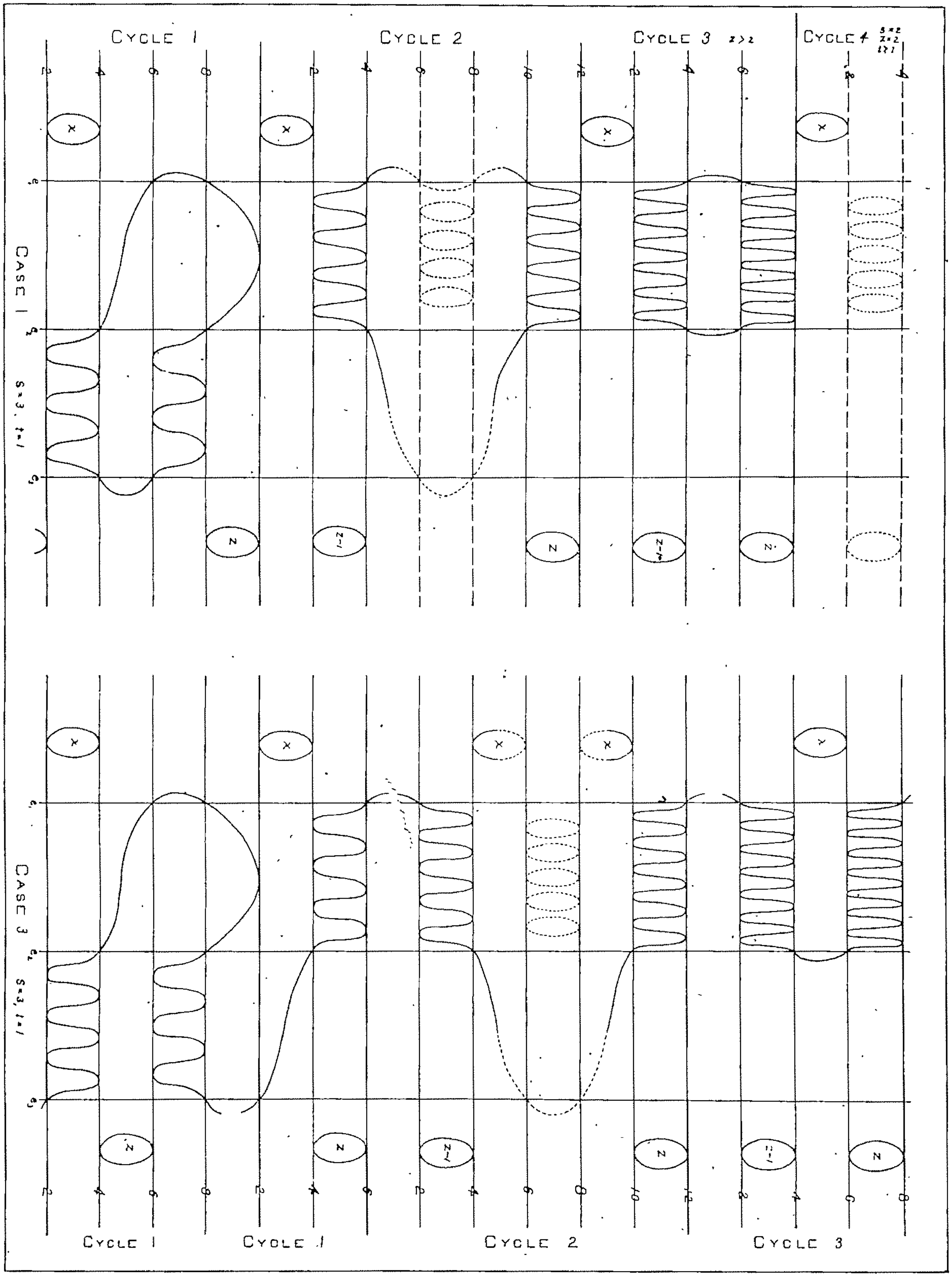
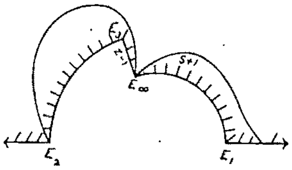

CYCLE 1
CYCLE 2
CYCLE 3 z>2
CYCLE 4 s=2 z=2 t>1
CASE 1
s=3, t=1
CASE 3
s=3, t=1
x
z
z-1
e1
e2
e3
2
4
6
8
10
12
CYCLE 1
CYCLE 1
CYCLE 2
CYCLE 3

PLATE IX



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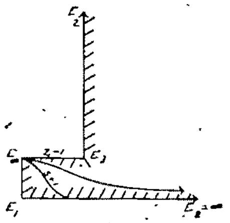

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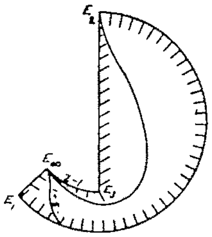

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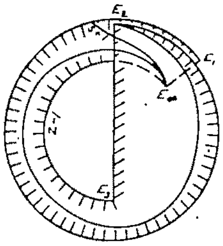

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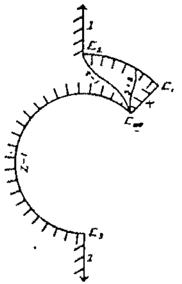

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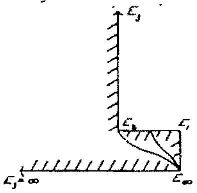

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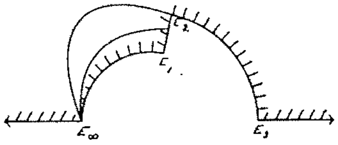

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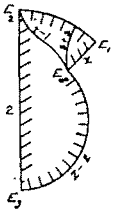

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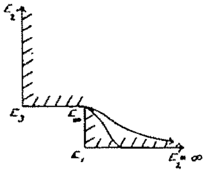

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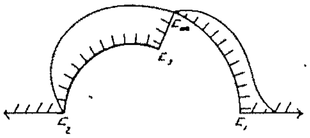

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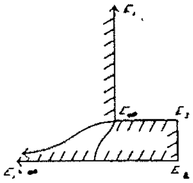

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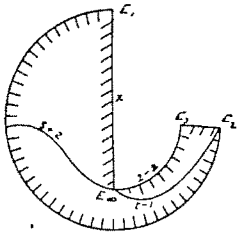

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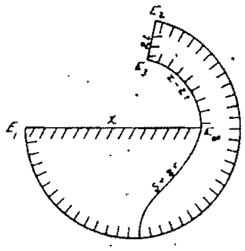
CYCLE 3



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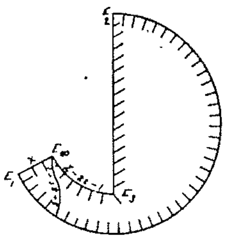

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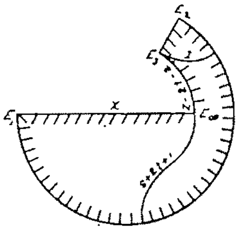

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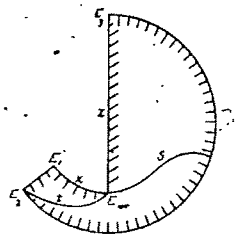
PLATE VI

CYCLE 1



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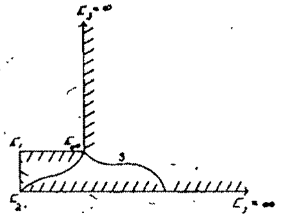

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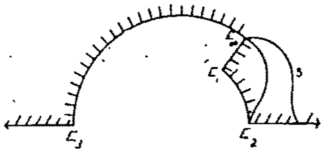

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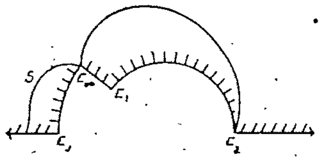

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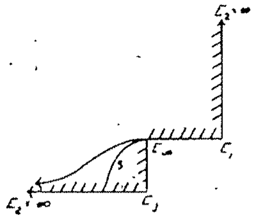

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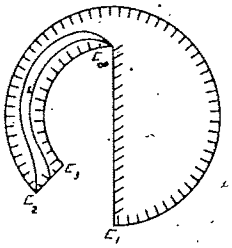

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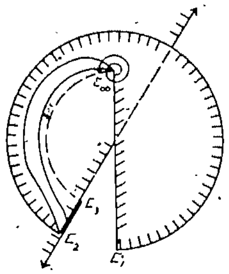

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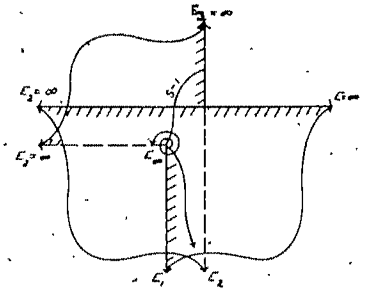

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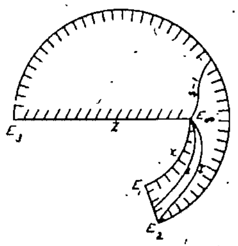

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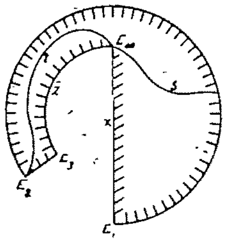
CYCLE 2



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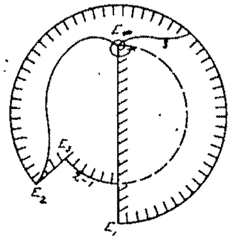

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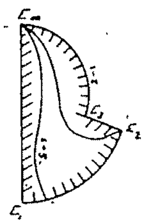

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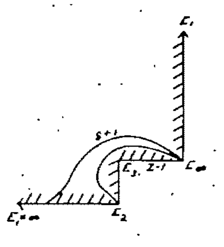

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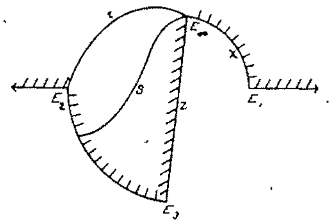
PLATE V

CYCLE I



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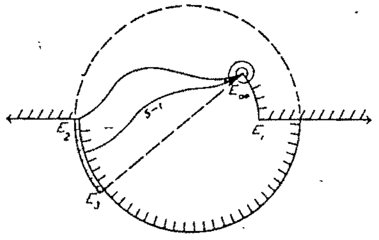

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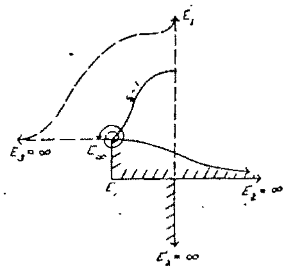

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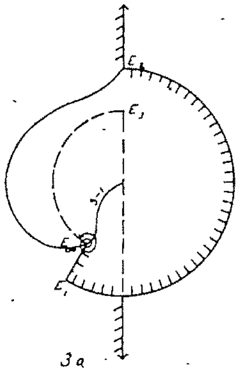

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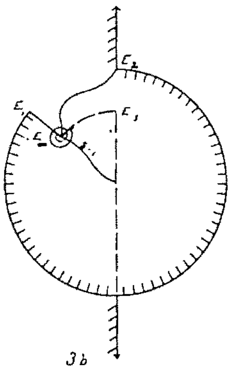

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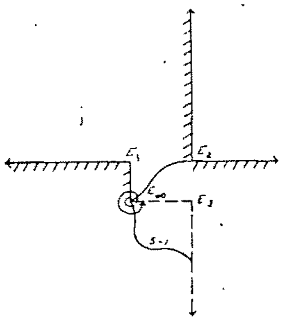

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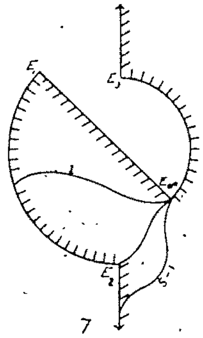

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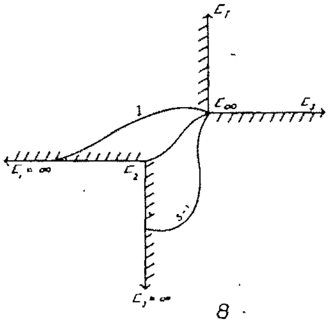

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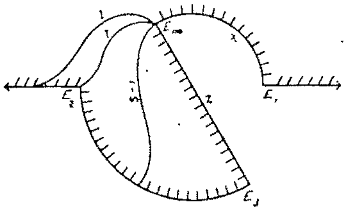

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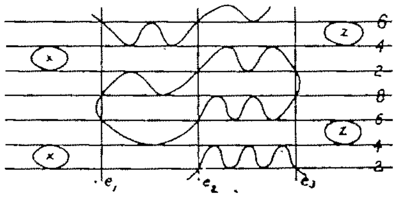

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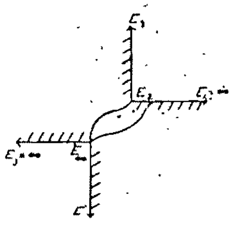
PLATE IV



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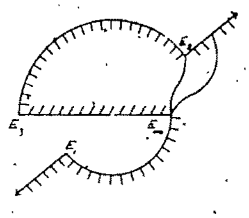

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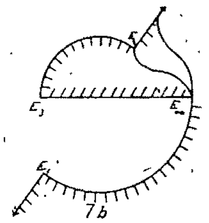

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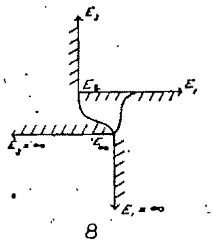

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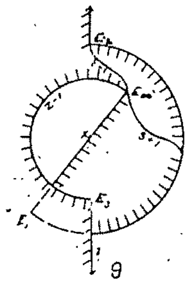

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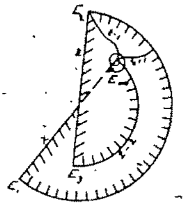

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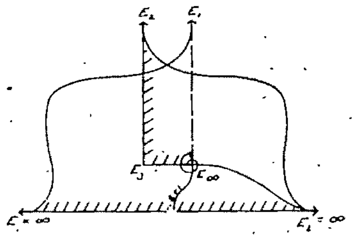

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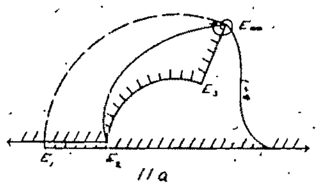

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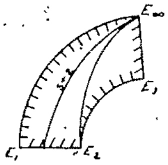

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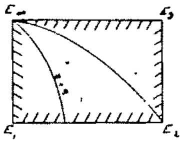

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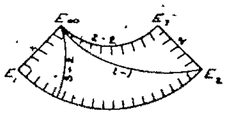

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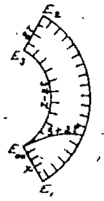

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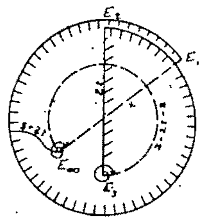

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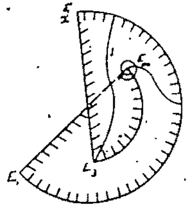

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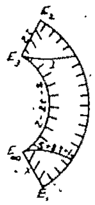

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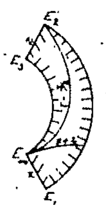

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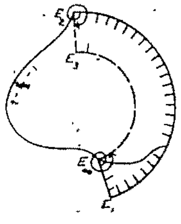

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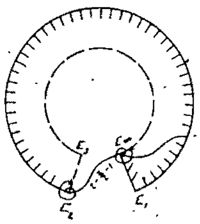

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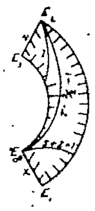
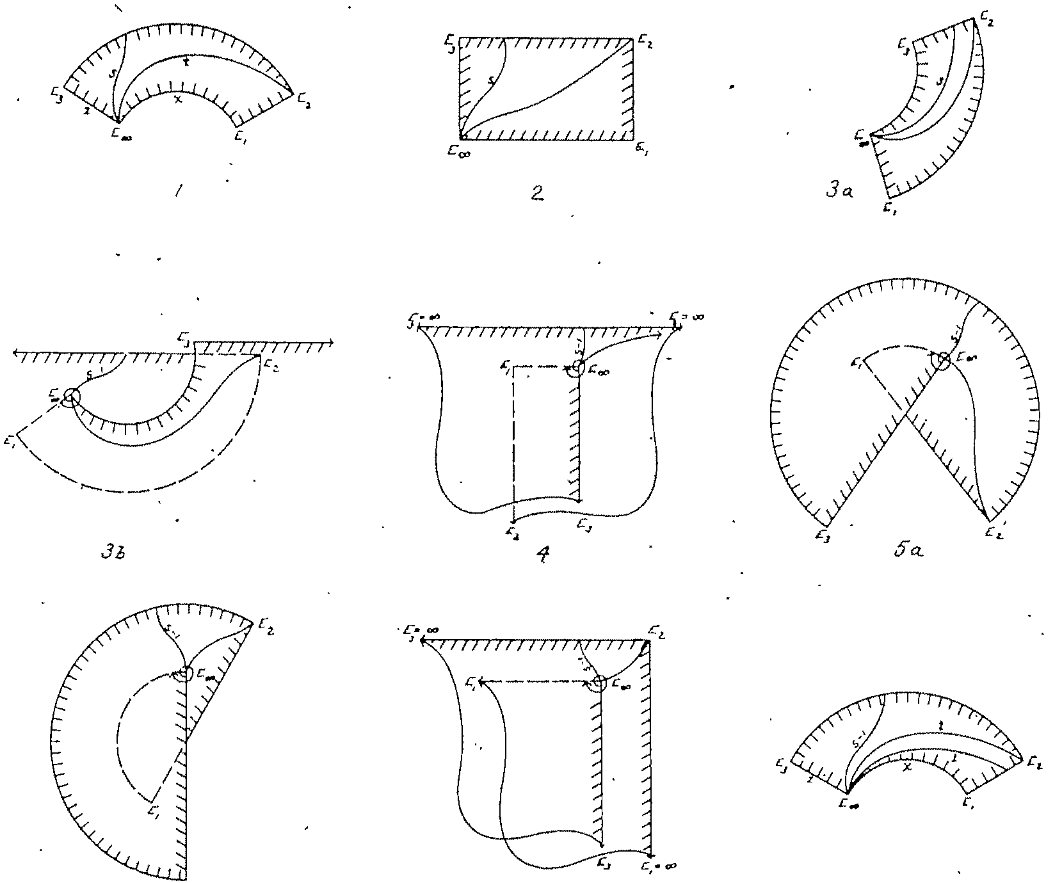


d

PLATE III

CYCLE 1

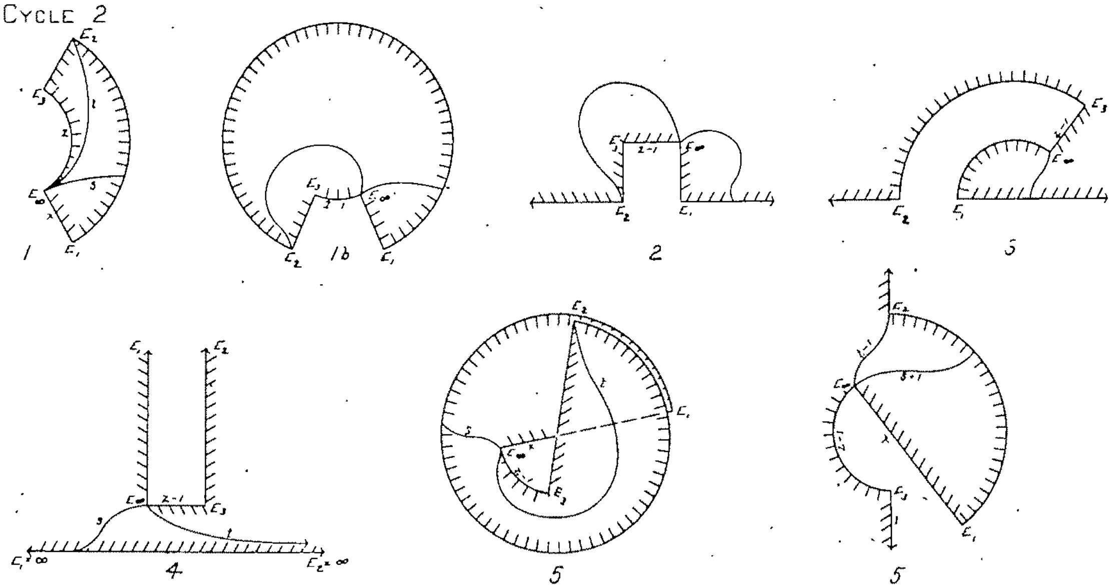


1 2 3a

3b 4 5a

5b 6 9

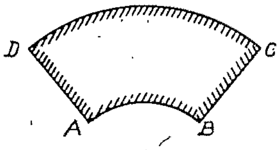
CYCLE 2



1 1b 2 3

4 5 5

PLATE II



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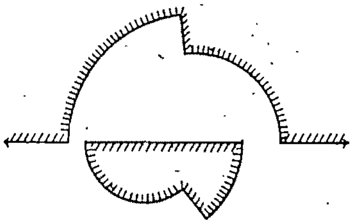

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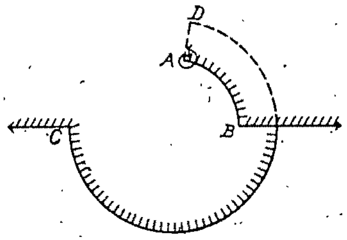

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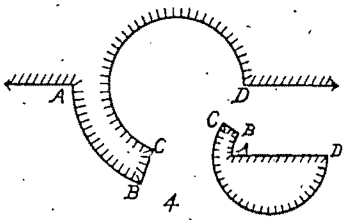

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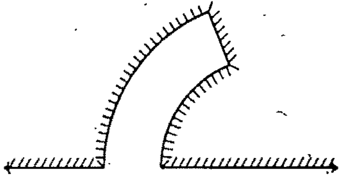

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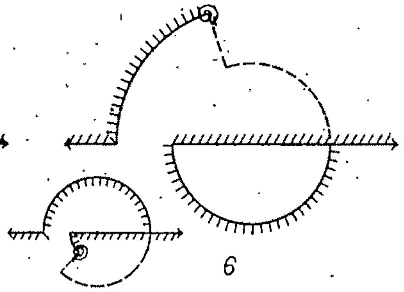

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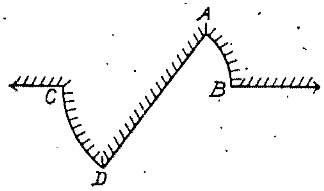

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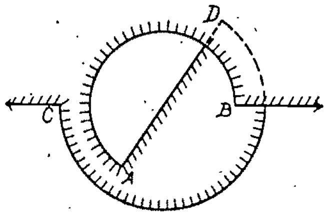
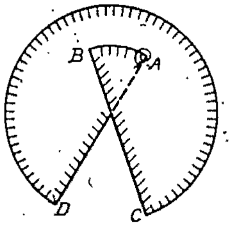


7(b)



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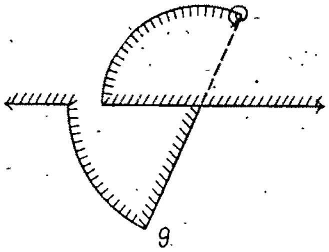

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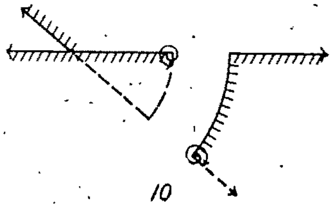

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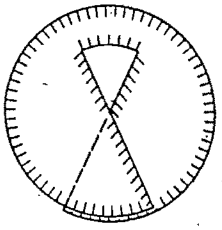

11

PLATE I

to ∞. Accordingly, for critical values of the parameter, the square root of the polynomial-product will have the form:

$$(1).\ (x-e_3)^{m_3+\frac{1}{2}}P_{\frac{n-1}{2}-m_3}, \qquad (2).\ (x-e_2)^{m_2-\frac{1}{2}}P_{\frac{n-1}{2}-m_2},$$

$$(3).\ (x-e_1)^{m_1+\frac{1}{2}}P_{\frac{n-1}{2}-m_1}, \qquad (4).\ P_{\frac{n-1}{2}-m_4=-\frac{n_\infty}{2}}.$$

Four classes of polynomials are thereby distinguished, and their total number is

$$2n-[m_1+m_2+m_3+m_4]+2=4t+2x+2y+2z+\begin{cases}3, \text{ case } 7,\\ 5, \text{ case } 8.\end{cases}$$

Of these only $\begin{cases}4t, \text{ case } 7,\\ 4(t+1), \text{ case } 8,\end{cases}$ belonging to the 1st and 3d classes can be imaginary.

It is noteworthy that in all 8 cases, with the single exception of case 6, the maximum number of imaginary polynomials is either $4t$ or $4(t+1)$, depending solely upon the number of diagonal attachments in the initial polygon.

WESLEYAN UNIVERSITY, *Aug.* 1898.

Note on Dfferential Invariants of a System of m Points by Projective Transformation.

By Edgar Odell Lovett.

As is well known the n^{th} order differential invariants of a geometrical configuration by an r parameter Lie group generated by the r independent infinitesimal transformations

$$U_1f, \;\; U_2f, \ldots, \; U_rf, \tag{1}$$

are found by integrating the complete system of partial differential equations

$$U_1^{(n)}f = 0, \;\; U_2^{(n)}f = 0, \ldots, \; U_r^{(n)}f = 0, \tag{2}$$

where the $U_i^{(n)}f$ are the so-called n^{th} extensions of the original transformations.

In particular if we seek a differential invariant of the second order of a system of m points in the plane by the general projective transformation group of that plane we have to find a solution of the complete system

$$\sum_{1}^{m}{}_{i} [X_1''f]_i = 0, \ldots \sum_{1}^{m}{}_{i}, [X_8''f]_i = 0, \tag{3}$$

where $[X_j''f]_i$ represents the result of substituting x_i, y_i for x, y in $X_j''f$, and $X_1''f, \ldots, X_8''f$ are the second extensions of the eight independent infinitesimal transformations which generate the projective group

$$p, \;\; q, \;\; xp, \;\; yp, \;\; xq, \;\; yq, \;\; x^2p + xyq, \;\; xyp + y^2q, \tag{4}$$

respectively, p being written for $\dfrac{\partial f}{\partial x}$ and q in place of $\dfrac{\partial f}{\partial y}$.

The second extension of the infinitesimal point transformation

$$X_if \equiv \xi_i(x, y)\,p + \eta_i(x, y)\,q, \tag{5}$$

is

$$X_i''f \equiv \xi_i p + \eta_i q + \eta_i' q' + \eta_i'' q'', \tag{6}$$

in which

$$\eta' \equiv \frac{d\eta}{dx} - y' \frac{d\xi}{dx}, \quad \eta'' \equiv \frac{d\eta'}{dx} - y'' \frac{d\xi}{dx}, \quad y' \equiv \frac{dy}{dx},$$
$$y'' \equiv \frac{d^2y}{dx^2}, \quad q' \equiv \frac{\partial f}{\partial y'}, \quad q'' \equiv \frac{\partial f}{\partial y''}. \qquad (7)$$

By means of these expressions the second extensions of the original eight infinitesimal projective transformations are found to be, respectively,

$$\left.\begin{array}{l} p,\, q,\;\; xp - y'q' - 2y''q'',\;\; yp - y'^2q' - 3y'y''q'',\;\; xq + q',\;\; yq + y'q' + y''q'' \\ x^2p + xyq + (y - xy')\,q' - 3xy''q'',\;\; xyp + y^2q + y'\,(y - xy')\,q' - 3xy'y''q''. \end{array}\right\} (8)$$

Then those functions $\phi\,(x_1, y_1, y_1', y_1'', \ldots\ldots, x_m, y_m, y_m', y_m'')$, which are differential invariants of the second order of a system of m points $(x_1, y_1, \ldots\ldots, x_m, y_m)$, are solutions of the complete system of equations

$$\left.\begin{array}{l} \sum\limits_1^m{}^i \frac{\partial\phi}{\partial x_i} = \sum\limits_1^m{}^i \frac{\partial\phi}{\partial y_i} = \sum\limits_1^m{}^i \left(x_i \frac{\partial\phi}{\partial x_i} - y_i' \frac{\partial\phi}{\partial y_i'} - 2y_i'' \frac{\partial\phi}{\partial y_i''}\right) \\ \qquad = \sum\limits_1^m{}^i \left(y_i \frac{\partial\phi}{\partial x_i} - y_i'^2 \frac{\partial\phi}{\partial y_i'} - 3y_i'y_i'' \frac{\partial\phi}{\partial y_i''}\right) = \sum\limits_1^m{}^i \left(x_i \frac{\partial\phi}{\partial y_i} + \frac{\partial\phi}{\partial y_i'}\right) = 0, \\ \sum\limits_1^m{}^i \left(y_i \frac{\partial\phi}{\partial y_i} + y_i' \frac{\partial\phi}{\partial y_i'} + y_i'' \frac{\partial\phi}{\partial y_i''}\right) \\ \qquad = \sum\limits_1^m{}^i \left\{x_i^2 \frac{\partial\phi}{\partial x_i} + x_iy_i \frac{\partial\phi}{\partial y_i} + (y_i - x_iy_i') \frac{\partial\phi}{\partial y_i'} - 3x_iy_i'' \frac{\partial\phi}{\partial y_i''}\right\} \\ \qquad = \sum\limits_1^m{}^i \left\{x_iy_i \frac{\partial\phi}{\partial x_i} + y_i^2 \frac{\partial\phi}{\partial y_i} + y_i'\,(y_i - x_iy_i') \frac{\partial\phi}{\partial y_i'} - 3x_iy_i'y_i'' \frac{\partial\phi}{\partial y_i''}\right\} = 0. \end{array}\right\} (9)$$

This simultaneous system has at least $4m - 8$ independent solutions; direct integration yields that m of these solutions are of the form

$$\sum_1^m{}^i \frac{y_k''}{y_i''} \left\{\frac{y_k - y_i - y_i'\,(x_k - x_i)}{y_k - y_i - y_k'\,(x_k - x_i)}\right\}^3, \quad k = 1,\, 2, \ldots\ldots, m. \quad (10)$$

In order to interpret these invariants geometrically, take m curves in the plane perfectly arbitrarily except that a curve is to pass through each of the m points of the system which we are studying; let ρ_i be the radius of curvature of the curve through (x_i, y_i) at (x_i, y_i); take any point (x_k, y_k) of the system of points and join it by straight lines to all the other points of the system; let θ_i be the angle at (x_k, y_k) between the normal to the curve through (x_k, y_k) and the line joining (x_k, y_k) to (x_i, y_i), and let ϕ_i be the angle between the latter line and the normal to the curve through (x_i, y_i); then the expressions (10) show that the m forms

$$\sum_{1}^{m}{}_{i}\frac{\rho_k \cos^3\theta_i}{\rho_i \cos^3\phi_i} \quad i \neq k, \ (k = 1, 2, \ldots, m) \quad (11)$$

are absolute invariants under the general projective group.

If, in particular, the m points lie on a straight line, these invariants reduce to the simpler forms

$$\sum_{1}^{m}{}_{i}\frac{\rho_k \cos^3\theta_k}{\rho_i \cos^3\theta_i}, \quad i \neq k, \quad (k = 1, 2, \ldots, m). \quad (12)$$

By means of the theorem of Reiss, stating that $\sum^{m}{}_{i}\rho_i^{-1}\cos^{-3}\theta_i$ is zero when a straight line cuts a curve of the m^{th} degree, we have that the value of the invariant is minus unity in case the m points lie on a straight line and a curve of the m^{th} degree simultaneously, and conversely.

Let the given system of points lie on a straight line and let the curves be so chosen that they are normal to the straight line at the points of the system. The invariants then become

$$\sum_{1}^{m-1}{}_{i}\frac{\rho_k}{\rho_i} \text{ or } \sum_{1}^{m}{}_{i}\frac{1}{\rho_i} = 0. \quad (13)$$

This includes as special cases the theorem that the ratio of the radii of curvature at corresponding points of two parallel curves is unaltered by projective transformation and also the theorem, given as new by Wölffing* but due to H. J. Stephen Smith,† that the ratio of the radii of curvature of two tangent curves at the point of tangency, is unaltered by projective transformation; it is obvious that the latter theorem appears without insisting that the curves be normal to the line.

In view of the fact that the general projective transformation group preserves its form under a transformation from point coordinates to line coordinates (the individual transformations are not invariant, but the group as a whole), corresponding theorems relative to the invariants of a system of m lines can be immediately written down from the above forms.

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* Wölffing, "Das Verhältniss der Krümmunsgsradien im Berührungspunkte zweier Curven," Zeitschrift für Mathematik und Physik, Bd. XXXVIII, 1893, pp. 287–249.

† H. J. Stephen Smith, "On the Focal Properties of Homographic Figures," Proc. London Mathematical Society, vol. II, pp. 196–248.

Consider a system of m points $(x_1, y_1, z_1, \ldots, x_m, y_m, z_m)$ in ordinary space. A differential invariant of this system by an infinitesimal point transformation

$$Vf \equiv \xi(x, y, z)\frac{\partial f}{\partial x} + \eta(x, y, z)\frac{\partial f}{\partial y} + \zeta(x, y, z)\frac{\partial f}{\partial z} \tag{14}$$

is a solution,

$$\psi(x_1, y_1, z_1, p_1, q_1, r_1, s_1, t_1, \ldots; \ldots, x_m, y_m, z_m, p_m, \ldots) \tag{15}$$

of the partial differential equation

$$\sum_1^m [V^{(n)}f]_i = 0,$$

where $V^{(n)}f$ is the n^{th} extension of Vf.

In order to determine the form of this extended transformation we proceed as follows:* Putting

$$p \equiv \frac{\partial z}{\partial x}, \quad q \equiv \frac{\partial z}{\partial y}, \quad r \equiv \frac{\partial^2 z}{\partial x^2}, \quad s \equiv \frac{\partial^2 z}{\partial x \partial y}, \quad t \equiv \frac{\partial^2 z}{\partial y^2}, \ldots, \tag{16}$$

the variation of the identity

$$dz \equiv pdx + qdy \tag{17}$$

gives

$$d\delta z \equiv \delta p \,.\, dx + \delta q \,.\, dy + pd\delta x + qd\delta y. \tag{18}$$

δx, δy, δz are given functions of x, y, z from (14), namely,

$$\delta x = \xi(x, y, z)\delta\varepsilon, \quad \delta y = \eta(x, y, z)\delta\varepsilon, \quad \delta z = \zeta(x, y, z)\delta\varepsilon, \tag{19}$$

where $\delta\varepsilon$ is an arbitrary infinitesimal of the first order. The identity (18) is to exist for all values of dx and dy, hence it breaks up into two equations for determining δp and δq, the variations of p and q by the infinitesimal transformation Vf. Calling these $\pi\delta\varepsilon$ and $\varkappa\delta\varepsilon$ respectively, the first extension of the transformation (14) is

$$V'f \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z} + \pi\frac{\partial f}{\partial p} + \varkappa\frac{\partial f}{\partial q}, \tag{20}$$

in which

$$\left.\begin{aligned} \pi &= \zeta x + p\zeta_z - p(\xi_x + p\xi_z) - q(\eta_x + p\eta_z), \\ \varkappa &= \zeta_y + q\zeta_z - p(\xi_y + q\xi_y) - q(\eta_y + q\eta_z). \end{aligned}\right\} \tag{21}$$

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* See Lie, "Vorlesungen über continuierliche Gruppen," bearbeitet und herausgegeben von Scheffers, Leipzig, 1893, pp. 709, 710.

Similarly the variation of the identities

$$dp \equiv r dx + s dy, \quad dq \equiv s dx + t dy \tag{22}$$

yields equations for the variations δr, δs, δt by the transformation Vf. The solution of these equations gives for the second extension of Vf,

$$V''f \equiv V'f + \rho \frac{\partial f}{\partial r} + \sigma \frac{\partial f}{\partial s} + \tau \frac{\partial f}{\partial t}, \tag{23}$$

in which

$$\left.\begin{aligned}
\rho &= \pi_x + p\pi_z + r\pi_p + s\pi_q - r(\xi_x + p\xi_z) - s(\eta_x + p\eta_z), \\
\sigma &= \pi_y + q\pi_z + s\pi_p + t\pi_q - r(\xi_y + q\xi_z) - s(\eta_y + q\eta_z) \\
&= \varkappa_x + p\varkappa_z + r\varkappa_p + s\varkappa_q - s(\xi_x + p\xi_z) - t(\eta_x + p\eta_z), \\
\tau &= \varkappa_y + q\varkappa_z + s\varkappa_p + t\varkappa_q - s(\xi_y + q\xi_z) - t(\eta_y + q\eta_z).
\end{aligned}\right\} \tag{24}$$

The higher extensions are computed in the same way. We have now the implements in hand by which to prosecute the study of the problems in space corresponding to those already discussed in the plane.

The formulæ (21) and (24) give the following forms to the second extensions of the fifteen independent infinitesimal transformations of the general projective group of ordinary space:

$$\left.\begin{aligned}
&\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}, \quad \frac{x\partial f}{\partial x} - p\frac{\partial f}{\partial p} - 2r\frac{\partial f}{\partial r} - s\frac{\partial f}{\partial s}, \\
&\frac{x\partial f}{\partial y} - q\frac{\partial f}{\partial p} - 2s\frac{\partial f}{\partial r} - t\frac{\partial f}{\partial s}, \quad \frac{x\partial f}{\partial z} + \frac{\partial f}{\partial p}, \\
&y\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial q} - r\frac{\partial f}{\partial s} - 2s\frac{\partial f}{\partial t}, \\
&y\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial q} - s\frac{\partial f}{\partial s} - 2t\frac{\partial f}{\partial t}, \quad y\frac{\partial f}{\partial z} + \frac{\partial f}{\partial q}, \\
&z\frac{\partial f}{\partial x} - p^2\frac{\partial f}{\partial p} - pq\frac{\partial f}{\partial q} - 3pr\frac{\partial f}{\partial r} - (2ps + qr)\frac{\partial f}{\partial s} - (2qs + pt)\frac{\partial f}{\partial t}, \\
&z\frac{\partial f}{\partial y} - pq\frac{\partial f}{\partial p} - q^2\frac{\partial f}{\partial q} - (2ps + qr)\frac{\partial f}{\partial r} - (2qs + pt)\frac{\partial f}{\partial s} - 3qt\frac{\partial f}{\partial t}, \\
&z\frac{\partial f}{\partial z} + p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} + r\frac{\partial f}{\partial r} + s\frac{\partial f}{\partial s} + t\frac{\partial f}{\partial t}, \\
&x^2\frac{\partial f}{\partial x} + xy\frac{\partial f}{\partial y} + zx\frac{\partial f}{\partial z} + (z - px - qy)\frac{\partial f}{\partial p} \\
&\qquad - (3rx + 2sy)\frac{\partial f}{\partial r} - (2sx + ty)\frac{\partial f}{\partial s} - tx\frac{\partial f}{\partial t},
\end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned}
&xy\frac{\partial f}{\partial x}+y^2\frac{\partial f}{\partial y}+yz\frac{\partial f}{\partial z}+(z-px-qy)\frac{\partial f}{\partial q}-rx\frac{\partial f}{\partial r}\\
&\qquad\qquad\qquad\qquad-(2sy+rx)\frac{\partial f}{\partial s}-(3ty+2sx)\frac{\partial f}{\partial t},\\
&zx\frac{\partial f}{\partial x}+yz\frac{\partial f}{\partial y}+z^2\frac{\partial f}{\partial z}+p(z-px-qy)\frac{\partial f}{\partial p}+q(z-px-qy)\frac{\partial f}{\partial q}\\
&\qquad-\{3prx+(qr+2ps)y\}\frac{\partial f}{\partial r}-\{(qr+2ps)x+(pt+2qs)y\}\frac{\partial f}{\partial s}\\
&\qquad-\{3qty+(pt+2qs)x\}\frac{\partial f}{\partial t}.
\end{aligned}\right\}\text{—Continued.}\ (25)$$

From these forms we write directly the simultaneous system of partial differential equations

$$\begin{aligned}
&\sum_{i=1}^{m}\frac{\partial\phi}{\partial x_i}=\sum_{i=1}^{m}\frac{\partial\phi}{\partial y_i}=\sum_{i=1}^{m}\frac{\partial\phi}{\partial z_i}\\
&\qquad\qquad=\sum_{i=1}^{m}\left(x_i\frac{\partial\phi}{\partial x_i}-p_i\frac{\partial\phi}{\partial p_i}-2r_i\frac{\partial\phi}{\partial r_i}-s_i\frac{\partial\phi}{\partial s_i}\right)=\ldots\ldots=0,
\end{aligned}\qquad(26)$$

to be satisfied by the second order differential invariants of the system of m points. The complete system is one of fifteen equations in $8m$ variables; hence there are at least $8m-15$ invariant functions. By integrating the system by successive substitutions, m of these invariants are found to have the following form:

$$\sum_{i=1}^{m}\frac{s_k^2-r_kt_k}{s_i^2-r_it_i}\left\{\frac{z_i-z_k-p_i(x_i-x_k)-q_i(y_i-y_k)}{z_i-z_k-p_k(x_i-x_k)-q_i(y_i-y_k)}\right\}^4,\qquad k=1,\ 2,\ \ldots\ldots,\ m.\qquad(27)$$

These invariants are susceptible of a geometrical interpretation quite analogous to that given in the plane. Take m surfaces in space perfectly arbitrarily chosen except that a surface is to pass through each of the m points of the system; let R_i and R'_i be the principal radii of curvature of the surface through (x_i, y_i, z_i) at (x_i, y_i, z_i); take any point (x_k, y_k, z_k) of the system of points and join it by straight lines to all the other points of the system; let θ_i be the angle at (x_k, y_k, z_k) between the normal to the surface through (x_k, y_k, z_k) and the line joining (x_k, y_k, z_k) to (x_i, y_i, z_i); and let ϕ_i be the angle between the latter line and the normal to the surface through (x_i, y_i, z_i); then the expressions (27) show that the m forms

$$\sum_1^m{}_i \frac{R_k R'_k \cos^4 \theta_i}{R_i R'_i \cos^4 \phi_i}, \qquad (28)$$

$$i \neq k, \quad k = 1, 2, \ldots, m$$

are absolute invariants by the general projective group.

When the m points lie on a straight line these m invariants reduce to the simpler forms

$$\sum_1^m{}_i \frac{R_k R'_k \cos^4 \theta_k}{R_i R'_i \cos^4 \theta_i}, \qquad (29)$$

$$i \neq k, \quad k = 1, 2, \ldots, m$$

If the m points lie on a straight line and a surface of the m^{th} degree simultaneously, we have

$$\sum_1^m{}_i \frac{1}{R_i R'_i \cos^4 \theta_i} = 0$$

as a generalization for ordinary space of the theorem of Reiss for the plane.

If the m points are collinear and the surfaces be so chosen that they are normal to the line the invariants become

$$\sum_1^m{}_i \frac{R_k R'_k}{R_i R_i}, \qquad (30)$$

$$i \neq k, \quad k = 1, 2, \ldots, m$$

This last result shows that to generalize the theorem of Smith relative to tangent curves and the theorem given relative to parallel curves it is only necessary to substitute "surface" for "curve" and "measure of curvature" for "radius of curvature." As a matter of fact there is nothing in the reasoning to forbid allowing the m points to fall together, and if we proceed to this limit from the case of collinearity and m surfaces normal to this line, the form (30) obtains in the limit and expresses an invariant relation by projective transformation in the curvatures of m surfaces tangent at a common point.

PRINCETON, N. J., 21 *April*, 1898.

Proof of Brioschi's Recursion Formula for the Expansion of the Even σ-Functions of Two Variables.

By Oskar Bolza.

In a note published in the Goettinger Nachrichten for 1890, p. 237, Brioschi has given, without proof, a recursion formula for the expansion of the even σ-functions of two variables, in which he makes use of a peculiar differential operator considerably easier to handle than the Aronhold process used by Wiltheiss* for the same purpose. He also gives the results of the application of this operator to the simultaneous concomitants of two cubics, and thus furnishes everything that is necessary for the actual computation of the successive terms of the expansion of the even σ-functions of two variables into power series.

In the following pages I propose to give a proof of these theorems of Brioschi's, since, as far as I know, no proof of them has ever been published.

§1. *The Partial Differential Equations for the Even σ-Functions of Two Variables.*

Let†

$$R(x) = \alpha_x^6 = \beta_x^6 = \ldots . \tag{1}$$

be a (non-homogeneous) sextic, and

$$H = (\alpha\beta)^2 \alpha_x^4 \beta_x^4, \quad i = (\alpha\beta)^4 \alpha_x^2 \beta_x^2, \quad A = (\alpha\beta)^6; \tag{2}$$

let, further,

$$R = \phi . \psi \tag{3}$$

be one of the ten possible decompositions of R into two cubic factors, and

$$\vartheta = (\phi, \psi)_1, \quad \Theta = (\phi, \psi)_2, \quad J = (\phi, \psi)_3, \tag{4}$$

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[*] Math. Annalen, Bd. 29, p. 272; Bd. 35, p. 433; Bd. 36, p. 134.

[†] Since we are using non-homogeneous variables, the symbols α_x, β_x, etc., stand for $\alpha_1 x + \alpha_2$, $\beta_1 x + \beta_2$, etc.

and write

$$\phi = \phi_0 x^3 + 3\phi_1 x^2 + 3\phi_2 x + \phi_3, \quad \psi = \psi_0 x^3 + 3\psi_1 x^2 + 3\psi_2 x + \psi_3$$
$$\vartheta = \vartheta_0 x^4 + 4\vartheta_1 x^3 + 6\vartheta_2 x^2 + 4\vartheta_3 x + \vartheta_4 .$$

Let, further,

$$y_2 = R(x) \tag{5}$$

and choose

$$w_1 = \int \frac{x\,dx}{2y}, \quad w_2 = \int \frac{dx}{2y} \tag{6}$$

for the two integrals of the first kind used in the formation of the σ-functions.

If, then, a denote one of the roots of ϕ, the following theorem holds:

Theorem I.

The σ-function belonging to the decomposition (3) *satisfies the partial differential equation*

$$\frac{\partial\sigma}{\partial a} = -\tfrac{1}{2}\sigma . \sum_{\alpha,\beta} \lambda_{\alpha\beta} u_\alpha u_\beta - \tfrac{9}{20} \frac{\Theta(a)}{R'(a)} \sigma - \sum_{\alpha,\beta} \varkappa_{\alpha\beta} u_\alpha \frac{\partial\sigma}{\partial u_\beta}$$
$$+ \tfrac{1}{4} \sum_{\alpha,\beta} \frac{\partial^2\sigma}{\partial u_\alpha \partial u_\beta} \frac{a^{4-\alpha-\beta}}{R'(a)}, \tag{A}$$

in which the summation indices take independently the values 1, 2 *and the quantities $\lambda_{\alpha\beta}$, $\varkappa_{\alpha\beta}$ are defined by the equations*

$$R'(a)(\lambda_{11}x^2 + 2\lambda_{12}x + \lambda_{22}) = -\tfrac{3}{4}\vartheta_x^2\vartheta_a^2 - \tfrac{3}{40}A(x-a)^2 \tag{7}$$

$$R'(a)(\varkappa_{11}x - \varkappa_{12}x\xi + \varkappa_{21} - \varkappa_{22}\xi)$$
$$= \tfrac{3}{2}\frac{\vartheta_0}{\phi_0}(3\phi_1 a^2 + 3\phi_2 a + \phi_3)(x-\xi) - \tfrac{3}{2}(3\vartheta_1 a^2 + 3\vartheta_2 a + \vartheta_3)(x-\xi) \tag{8}$$
$$+ \tfrac{3}{2}\vartheta_a^2\vartheta_x\vartheta_\xi + \tfrac{9}{10}\Theta_x\Theta_a(a-\xi) + \tfrac{1}{4}J(x-a)(\xi-a).$$

Proof: In a previous paper, "The Partial Differential Equations of the Hyperelliptic Θ- and σ-Functions,"* I have given a proof of (A) with the following defining equations for the $\varkappa_{\alpha\beta}$'s and $\lambda_{\alpha\beta}$'s:

$$\Lambda(x, \xi) \equiv \tfrac{1}{4}[\lambda_{11}x . + \lambda_{12}(x+\xi) + \lambda_{22}]$$
$$= \tfrac{1}{4}\left(\frac{1}{x-a} + \frac{1}{\xi-a}\right)\frac{F(x, \xi)}{(x-\xi)^2} - \tfrac{1}{2}\frac{F(x, a)F(\xi, a)}{R'(a)(x-a)^2(\xi-a)^2} + \tfrac{1}{2}\frac{\frac{\partial}{\partial a}F(x, \xi)}{(x-\xi)^3} \tag{9}$$

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* American Journal of Mathematics, vol. XXI, p. 107, equations.

and

$$K(x,\,\xi) \equiv \varkappa_{11}x - \varkappa_{12}x\xi + \varkappa_{21} - \varkappa_{22}\xi = \tfrac{1}{2}\,\frac{(x-\xi)}{x-a} - \frac{(a-\xi)\,F(x,\,a)}{R'(a)(x-a)^2}, \qquad (10)$$

where $F(x,\,\xi) = \alpha_x^3\,\alpha_\xi^3$ is the third polar of $R(x)$ with respect to ξ.

It only remains therefore to show that (9) and (10) are equivalent to (7) and (8).

a). The expression for $\Lambda(x,\,\xi)$ can be transformed as follows:

Notice first that $\Lambda(x,\,\xi)$ is the first polar of $\Lambda(x,\,x)$ with respect to ξ; but by letting $\xi = x$ and making use of the expansion

$$F(x,\,\xi) = R(x) + \tfrac{1}{2}\,R'(x)(\xi - x) + \tfrac{1}{10}\,R''(x)(\xi - x)^2 + \dots.$$

we obtain

$$R'(a)\,\Lambda(x,\,x) = -\tfrac{1}{4}\left(\frac{F(x,\,a)}{(x-a)^2}\right)^2 + \tfrac{3}{20}\,\frac{R(x)\,R'(a) - R(a)\,R'(x)}{(x-a)^3} - \tfrac{1}{40}\,\frac{R'(a)\,R'(x)}{(x-a)^2},$$

where the zero term $R(a)\,R'(x)$ has been added for symmetry.

But since

$$R(x) = \alpha_x^5\alpha_1 \,.\, x + \alpha_x^5\alpha_2 = \tfrac{1}{6}\,xR'(x) + \alpha_x^5\alpha_2,$$

we have

$$R(x)\,R'(a) - R(a)\,R'(x) = \tfrac{1}{6}\,(x-a)\,R'(x)\,R'(a) - 6\,(\alpha\beta)\,\alpha_x^5\beta_a^5,$$

and therefore

$$R'(a)\,\Lambda(x,\,x) = -\frac{1}{(x-a)^4}\left[\tfrac{1}{2}\,\alpha_x^3\alpha_a^3 \,.\, \beta_x^3\beta_a^3 + \tfrac{9}{10}\,(x-a)\,.\,(\alpha\beta)\,\alpha_x^5\beta_a^5\right].$$

Now observe that

$$2\alpha_x^3\alpha_a^3 \,.\, \beta_x^3\beta_a^3 = \alpha_x^6\beta_a^6 + \alpha_a^6\beta_x^6 - (\alpha_x^3\beta_a^3 - \alpha_a^3\beta_x^3)^2$$

and apply Clebsch–Gordon's expansion, which furnishes:

$$\begin{aligned}
&(\alpha\beta)^2(\alpha_x^4\beta_a^4 + \alpha_a^4\beta_x^4)\\
&\qquad = 2H_x^4H_a^4 + \tfrac{24}{7}\,i_x^2 i_a^2\,(x-a)^2 + \tfrac{2}{5}\,A\,(x-a)^4\\
&\qquad = 2\,(\alpha\beta)^2(\alpha_x^3\alpha_a\beta_x\beta_a^3 + \alpha_a^3\alpha_x\beta_a\beta_x^3)\;4\,H_x^4H_a^4 + \tfrac{6}{7}\,i_x^2 i_a^2\,(x-a)^2 - \tfrac{1}{5}\,A\,(x-a)^4\\
&\qquad = (\alpha\beta)\,\alpha_x^5\beta_a^5\;\tfrac{5}{2}\,H_x^4H_a^4\,(x-a) + \tfrac{25}{14}\,i_x^2 i_a^2\,(x-a)^3 + \tfrac{1}{6}\,A\,(x-a)^5.
\end{aligned}$$

The result is equation (7).

b). Similarly the expression for $K(x,\,\xi)$ can be transformed as follows: By Clebsch–Gordon's expansion we have

$$\phi_a^3\psi_x^3 = \alpha_x^3\alpha_a^3 - \tfrac{3}{2}\,\vartheta_x^2\vartheta_a^2\,(x-a) + \tfrac{9}{10}\,\Theta_x\Theta_a\,(x-a)^2 - \tfrac{1}{4}\,J\,(x-a)^3.$$

Hence since

$$\phi_a^3 = 0,$$

$$\frac{F(x, a)}{x-a} = \frac{3}{2}\vartheta_x^2\vartheta_a^2 - \frac{9}{10}\Theta_x\Theta_a(x-a) + \frac{1}{4}J(x-a)^3.$$

Further,

$$\frac{\phi(x)\psi(a)}{x-a} = \frac{\phi(x)\psi(a) - \phi(a)\psi(x)}{x-a} = 3\vartheta_x^2\vartheta_a^2 + \frac{1}{2}J(x-a)^2,$$

and by making $x = a$:

$$R'(a) = \phi'(a)\psi(a) = 3\vartheta_a^4.$$

Thus we obtain

$$R'(a)K(x, \xi) \\ = \frac{3}{2}\frac{(x-\xi)\vartheta_a^4 - (a-\xi)\vartheta_a^2\vartheta_x^2}{x-a} + \frac{9}{10}\Theta_x\Theta_a(a-\xi) + \frac{1}{4}J(x-a)(\xi-a),$$

which, after performing the division by $x-a$, reduces to

$$R'(a)K(x, \xi) \\ = -\frac{3}{2}\vartheta_a^3\vartheta_1(x-\xi) + \frac{3}{2}\vartheta_a^2\vartheta_x\vartheta_\xi + \frac{9}{10}\Theta_x\Theta_a(a-\xi) + \frac{1}{4}J(x-a)(\xi-a).$$

And if we reduce the degree of the right-hand side in a by means of the equation $\phi(a) = 0$, we obtain (8).

Thus Theorem I is proved.

If b denote a root of ψ, the expression for $\frac{\partial\sigma}{\partial b}$ can be derived from (A) by simply writing b instead of a and interchanging ϕ and ψ, which operation changes the signs of ϑ and J, but leaves Θ, i, A unchanged. We shall refer to the differential equation thus obtained as equation (A′).

§2.—*Brioschi's Differential Operator and the Recursion Formula for the Expansion of* $\sigma_{\phi\psi}(u_1, u_2)$.

Multiply equation (A) by

$$3\vartheta_u^2\vartheta_a^2 + \frac{1}{2}J(u-a)^2 = \frac{\phi(u)\psi(a)}{u-a},$$

u being an arbitrary new variable, and sum with respect to the three roots of ϕ (notation Σ). Similarly, multiply (A′) by

$$-3\vartheta_u^2\vartheta_b^2 - \frac{1}{2}J(u-b)^2 = \frac{\psi(u)\phi(b)}{u-b}$$

and sum with respect to the three roots of ψ (notation Σ'). Add the two equations thus obtained and finally put $u = \frac{u_1}{u_2}$ and multiply by u_2^2.

The left-hand side becomes*

$$(3\vartheta_u^2\vartheta_1^2 + \tfrac{1}{2}Ju_2^2)\left(\Sigma a^2\frac{\partial\sigma}{\partial a} - \Sigma' b^2\frac{\partial\sigma}{\partial b}\right) + 2\,(3\vartheta_u^2\vartheta_1\vartheta_2 - \tfrac{1}{2}Ju_1u_2)\left(\Sigma a\frac{\partial\sigma}{\partial a} - \Sigma' b\frac{\partial\sigma}{\partial b}\right)$$
$$+\ (3\vartheta_u^2\vartheta_2^2 + \tfrac{1}{2}Ju_1^2)\left(\Sigma\frac{\partial\sigma}{\partial a} - \Sigma'\frac{\partial\sigma}{\partial b}\right).$$

The right-hand side of (A) is an integral function of the second degree of a, say $q(a)$, divided by $R'(a)$, and the right-hand side of (A′) is an integral function of the second degree of b, say $\bar{q}(b)$, divided by $R'(b)$. But

$$\Sigma\,\frac{\phi(u)\psi(a)}{u-a}\cdot\frac{q(a)}{R'(a)} = \Sigma\,\frac{q(a)}{\phi'(a)}\,\frac{\phi(u)}{u-a} = q(u),$$
$$\Sigma'\frac{\psi(u)\phi(b)}{u-b}\cdot\frac{\bar{q}(b)}{R'(b)} = \Sigma\,\frac{\bar{q}(b)}{\psi'(b)}\,\frac{\psi(u)}{u-b} = \bar{q}(u).$$

Hence the result of the above-described operation is, for the right-hand side,

$$q(u) + \bar{q}(u),$$

and if we observe that†

$$\phi_0\psi_1 - \phi_1\psi_0 = \vartheta_0,$$
$$\phi_0\psi_2 - \phi_2\psi_0 = 2\vartheta_1,$$
$$\phi_0\psi_3 - \phi_3\psi_0 = 3\vartheta_2 + \tfrac{1}{2}J,$$

and

$$\frac{3\vartheta_0}{\phi_0\psi_0} = s - s',$$

where

$$s = \Sigma a, \quad s' = \Sigma' b, \tag{11}$$

this reduces to

$$3i_u^4.\sigma - \tfrac{3}{10}\Theta_u^2.\sigma - \tfrac{3}{5}\Theta_u^2\left(u_1\frac{\partial\sigma}{\partial u_1} + u_2\frac{\partial\sigma}{\partial u_2}\right)$$
$$+\ \tfrac{1}{2}\left(\frac{\partial^2\sigma}{\partial u_1^2}u_1^2 + 2\,\frac{\partial^2\sigma}{\partial u_1\,\partial u_2}u_1u_2 + \frac{\partial^2\sigma}{\partial u_2^2}u_2^2\right)$$
$$+\ \tfrac{1}{2}\,(s-s')(3\vartheta_u^2\vartheta_1^2 + \tfrac{1}{2}J)\left(u_1\frac{\partial\sigma}{\partial u_1} + u_2\frac{\partial\sigma}{\partial u_2}\right).$$

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*It is hardly necessary to say that here ϑ_1, ϑ_2 are symbols, and that $\vartheta_u = \vartheta_1u_1 + \vartheta_2u_2$.

†Here ϑ_0, ϑ_1, ϑ_2 are of course coefficients of ϑ.

Transposing the last term, we obtain the

Theorem II.

If the three operators $G_0(f)$, $G_1(f)$, $G_2(f)$ *are defined by the equations*

$$\left.\begin{aligned} G_0(f) &= \Sigma \frac{\partial f}{\partial a} - \Sigma' \frac{\partial f}{\partial b}, \\ G_1(f) &= \Sigma a \frac{\partial f}{\partial a} - \Sigma' b \frac{\partial f}{\partial b}, \\ G_2(f) &= \Sigma a^2 \frac{\partial f}{\partial a} - \Sigma' b^2 \frac{\partial f}{\partial b} - \frac{s - s'}{2}\left(u_1 \frac{\partial f}{\partial u_1} + u_2 \frac{\partial f}{\partial u_2}\right), \end{aligned}\right\} \tag{12}$$

where $s = \Sigma a, \quad s' = \Sigma' b,$

and the operator $D(f)$ *by*

$$\begin{aligned} D(f) = (6\vartheta_u^2\vartheta_1^2 + Ju_2^2)\, G_2(f) + 2(6\vartheta_u^2\vartheta_1\vartheta_2 - Ju_1u_2)\, G_1(f) \\ + (6\vartheta_u^2\vartheta_2^2 + Ju_1^2)\, G_0(f), \end{aligned} \tag{13}$$

the function $\sigma_{\phi\psi}(u_1 u_2)$ *satisfies the partial differential equation*

$$\begin{aligned} D(\sigma) = 6i_u^4 . \sigma - \tfrac{9}{5}\Theta_u^2 . \sigma - \tfrac{18}{5}\Theta_u^2 \left(u_1 \frac{\partial \sigma}{\partial u_1} + u_2 \frac{\partial \sigma}{\partial u_2}\right) \\ + \left(u_1^2 \frac{\partial^2 \sigma}{\partial u_1^2} + 2u_1 u_2 \frac{\partial^2 \sigma}{\partial u_1 \partial u_2} + u_2^2 \frac{\partial^2 \sigma}{\partial u_2^2}\right). \end{aligned} \tag{B}$$

We have chosen the two integrals of the first kind, w_1, w_2, in accordance with Wiltheiss and Brioschi; if, instead, we had taken $\bar{w}_1 = cw_1$, $\bar{w}_2 = cw_2$, we would have reached a slightly different result, viz. the first and last terms on the right would be changed into

$$\frac{1}{c^2} . 6i_u^4\sigma \text{ and } c^2 . \left(u_1^2 \frac{\partial^2 \sigma}{\partial u_1^2} + 2u_1 u_2 \frac{\partial^2 \sigma}{\partial u_1 \partial u_2} + u_2^2 \frac{\partial^2 \sigma}{\partial u_2^2}\right),$$

as follows from the formulæ* for the passage from one canonical system of integrals to another. Klein, in his paper on "Hyperelliptische Sigmafunctionen" (Math. Annalen, Bd. 27), chooses

$$\bar{w}_1 = -\int \frac{x\,dx}{y}, \quad \bar{w}_2 = -\int \frac{dx}{y},$$

hence $c = -2$.

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* Bolza, "On Weierstrass' Systems of Hyperelliptic Integrals of the First and Second Kind," Chicago International Mathematical Congress Papers, p. 8.

Now let
$$\sigma(u_1, u_2) = 1 + \frac{S_1}{2!} + \frac{S_2}{4!} + \frac{S_3}{6!} + \dots \qquad (14)$$

be the expansion of $\sigma_{\phi\psi}$ according to powers of u_1, u_2, $\frac{S_n}{2n!}$ denoting the aggregate of the terms of dimension $2n$.

Substitute this series in (B) and equate the terms of dimension $2n$ on both sides. Remembering that

$$u_1 \frac{\partial S_{n-1}}{\partial u_1} + u_2 \frac{\partial S_{n-1}}{\partial u_2} = (2n-2)\, S_{n-1}$$

and
$$u_1^2 \frac{\partial^2 S_n}{\partial u_1^2} + 2u_1 u_2 \frac{\partial^2 S_n}{\partial u_1 \partial u_2} + u_2^2 \frac{\partial^2 S_n}{\partial u_2^2} = 2n\,.\,(2n-1)\, S_n,$$

we obtain
$$D(S_{n-1}) = 12\,(n-1)(2n-3)\, i_u^4 S_{n-2} - \tfrac{2}{5}\,(4n-3)\, \Theta_u^2 S_{n-1} + S_n\,.$$

This furnishes for $n = 1$
$$S_1 = \tfrac{2}{5}\, \Theta_u^2 \qquad (15)$$

and thus we reach Brioschi's theorem:

Theorem III.

The successive terms of the expansion of $\sigma_{\phi\psi}(u_1, u_2)$ *into a power series*

$$\sigma_{\phi\psi}(u_1, u_2) = 1 + \frac{S_1}{2!} + \frac{S_2}{4!} + \frac{S_3}{6!} + \dots$$

are determined by the recursion formula

$$S_n = D(S_{n-1}) + (4n-3)\, S_1 S_{n-1} - 12\,(n-1)(2n-3)\, i_u^4 S_{n-2} \qquad (C)$$

where the operator D is defined by (13) *and*

$$S_1 = \tfrac{2}{5}\,(\phi\psi)^2 \phi_u \psi_u\,, \quad i_u^4 = (\alpha\beta)^4 \alpha_u^2 \beta_u^2\,.$$

Corollary: The covariant i is expressible in terms of the simultaneous concomitants of ϕ and ψ as follows:*

$$i = \tfrac{6}{25}\, \Theta^2 + \tfrac{3}{10}\, \Delta\nabla - \tfrac{1}{5}\, J\vartheta\,, \qquad (16)$$

in the notation of Clebsch, "Binaerie Formen," §61.

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[*] Proved by Caporali, Sul sistema di due forme binarie cubiche, §8, Rendiconto della R. Accademia di Napoli, 1888.

§3. *Reduction of Brioschi's Operator $D(f)$ to an Aronhold Process.*

From

$$3\frac{\partial\phi_1}{\partial a} = -\phi_0$$
$$3\frac{\partial\phi_2}{\partial a} = -(\phi_0 a + 3\phi_1)$$
$$\frac{\partial\phi_3}{\partial a} = -(\phi_0 a^2 + 3\phi_1 a + 3\phi_2)$$

follows

$$\Sigma \frac{\partial f}{\partial a} = -\phi_0\frac{\partial f}{\partial\phi_1} - 2\phi_1\frac{\partial f}{\partial\phi_2} - 3\phi_2\frac{\partial f}{\partial\phi_3}$$
$$\Sigma a\frac{\partial f}{\partial a} = \phi_1\frac{\partial f}{\partial\phi_1} + 2\phi_2\frac{\partial f}{\partial\phi_2} + 3\phi_3\frac{\partial f}{\partial\phi_3}$$
$$\Sigma a^2\frac{\partial f}{\partial a} = 3\phi_1\frac{\partial f}{\partial\phi_0} + 2\phi_2\frac{\partial f}{\partial\phi_1} + \phi_3\frac{\partial f}{\partial\phi_2} - \frac{3\phi_1}{\phi_0}\left(\phi_0\frac{\partial f}{\partial\phi_0} + \phi_1\frac{\partial f}{\partial\phi_1} + \phi_2\frac{\partial f}{\partial\phi_2} + \phi_3\frac{\partial f}{\partial\phi_3}\right).$$

Hence if f be a homogeneous function of $\phi_0, \phi_1, \phi_2, \phi_3$ of degree ν and of u_1, u_2 of degree m, we have

$$\Sigma a^2\frac{\partial f}{\partial a} - \frac{s}{2}\left(u_1\frac{\partial f}{\partial u_1} + u_2\frac{\partial f}{\partial u_2}\right) = s\left(\nu - \frac{m}{2}\right)f + 3\phi_1\frac{\partial f}{\partial\phi_0} + 2\phi_2\frac{\partial f}{\partial\phi_1} + \phi_3\frac{\partial f}{\partial\phi_3}.$$

We are going to apply the operator D successively to $S_1, S_2, S_3, \ldots$; but it is easily seen from (C) that S_n is of degree $2n$ in u_1, u_2, of degree n in the ϕ_i's and of degree n in the ψ_i's. Hence we are only dealing with functions f for which

$$\nu - \frac{m}{2} = 0,$$

and for such functions we have

$$\Sigma a^2\frac{\partial f}{\partial a} - \frac{s}{2}\left(u_1\frac{\partial f}{\partial u_1} + u_2\frac{\partial f}{\partial u_2}\right) = 3\phi_1\frac{\partial f}{\partial\phi_0} + 2\phi_2\frac{\partial f}{\partial\phi_1} + \phi_3\frac{\partial f}{\partial\phi_2}.$$

Analogous formulæ hold for

$$\Sigma'\frac{\partial f}{\partial b}, \quad \Sigma' b\frac{\partial f}{\partial b}, \quad \Sigma' b^2\frac{\partial f}{\partial b}$$

and we thus obtain for $G_0(f)$, $G_1(f)$, $G_2(f)$ expressions which are ***homogeneous and linear*** in

$$\frac{\partial f}{\partial \phi_0}, \quad \frac{\partial f}{\partial \phi_1}, \quad \frac{\partial f}{\partial \phi_2}, \quad \frac{\partial f}{\partial \phi_3}, \quad \frac{\partial f}{\partial \psi_0}, \quad \frac{\partial f}{\partial \psi_1}, \quad \frac{\partial f}{\partial \psi_2}, \quad \frac{\partial f}{\partial \psi_3}.$$

For their further simplification we may therefore write symbolically:

$$\frac{\partial f}{\partial \phi_0} = x_1^3, \quad \frac{\partial f}{\partial \phi_1} = 3x_1^2x_2, \quad \frac{\partial f}{\partial \phi_2} = 3x_1x_2^2, \quad \frac{\partial f}{\partial \phi_3} = x_2^3$$
$$\frac{\partial f}{\partial \psi_0} = y_1^3, \quad \frac{\partial f}{\partial \psi_1} = 3y_1^2y_2, \quad \frac{\partial f}{\partial \psi_2} = 3y_1y_2^2, \quad \frac{\partial f}{\partial \psi_3} = y_2^3,$$

perform the reductions in the symbolical form and in the final result resubstitute $\frac{\partial f}{\partial \phi_0}$, etc., for the symbols.

Thus
$$\begin{aligned} G_0(f) &= -3\left[x_2\phi_x^2\phi_1 - y_2\psi_y^2\psi_1\right], \\ G_1(f) &= 3\left[x_2\phi_x^2\phi_2 - y_2\psi_y^2\psi_2\right], \\ G_2(f) &= 3\left[x_1\phi_x^2\phi_2 - y_1\psi_y^2\psi_2\right]. \end{aligned}$$

Since the degree ν of f in the ϕ_i's is supposed to be the same as in the ψ_i's, we have

$$\phi_x^3 = \psi_y^3 = \nu f,$$

and $G_1(f)$ may therefore be written

$$\begin{aligned} G_1(f) &= 3\left[\phi_x^3 - \psi_y^3 - x_1\phi_x^2\phi_1 + y_1\psi_y^2\psi_1\right] \\ &= 3\left[-x_1\phi_x^2\phi_1 + y_1\psi_y^2\psi_1\right], \end{aligned}$$

hence

$$2G_1(f) = 3\left[(x_2\phi_2 - x_1\phi_1)\,\phi_x^2 - (y_2\psi_2 - y_1\psi_1)\,\psi_y^2\right].$$

Now if we put

$$\Phi_z^2 = 6\vartheta_u^2\vartheta_z^2 + J(uz)^2 = c_{11}z_1^2 + 2c_{12}z_1z_2 + c_{22}z_2^2$$

we may write

$$\begin{aligned} D(f) &= c_{11}G_2(f) + 2c_{12}G_1(f) + c_{22}G_0(f) \\ &= 3\phi_x^2\left[c_{11}x_1\phi_2 + c_{12}(x_2\phi_2 - x_1\phi_1) - c_{22}x_2\phi_1\right] \\ &\quad - 3\psi_y^2\left[c_{11}y_1\psi_2 + c_{12}(y_2\psi_2 - y_1\psi_1) - c_{22}y_2\psi_1\right], \end{aligned}$$

or
$$D(f) = 3\,(\Phi\phi)\,\Phi_x\phi_x^2 - 3\,(\Phi\psi)\,\Phi_y\psi_y^2.$$

We thus reach the

Theorem IV.

If
$$\Phi_x^2 = 6\vartheta_u^2\vartheta_x^2 + J(ux)^2 \tag{17}$$

and

$$\begin{aligned} 3\,(\Phi\phi)\,\Phi_x\phi_x^2 &= M_0x_1^3 + 3M_1x_1^2x_2 + 3M_2x_1x_2^2 + M_3x_2^3 = M_x^3 \\ -3\,(\Phi\psi)\,\Phi_x\psi_x^2 &= N_0x_1^3 + 3N_1x_1^2x_2 + 3N_2x_1x_2^2 + N_3x_2^3 = N^3 \end{aligned} \tag{18}$$

then Brioschi's operator $D(f)$ is equivalent to the following Aronhold process:

$$D(f) = \sum_{i=0}^{3} M_i \frac{\partial f}{\partial \phi_i} + \sum_{i=0}^{3} N_i \frac{\partial f}{\partial \psi_i}. \tag{D}$$

Corollary: Hence follows the rule:

To obtain $D(f)$, replace in f the coefficients ϕ_i and ψ_i by the corresponding coefficients $\phi_i + \lambda M_i$ and $\psi_i + \lambda N_i$ respectively and expand according to powers of λ. $D(f)$ will be the coefficient of λ in this expansion.

In applying this rule to a function of u_1, u_2:

$$f = f_u^m,$$

we first compute

$$D(f_x^m)$$

and in the result put $x = u$. This precaution is necessary, if we wish to use symbolical methods, since u_1, u_2 enter into the coefficients of M and N as well as into the function f_u^m.

§4. *Effect of the Operator D upon the Simultaneous Concomitants of ϕ and ψ.*

We now proceed to determine the effect of the operator D upon the simultaneous concomitants of ϕ and ψ as far as they appear in the successive terms $S_1, S_2, S_3, \ldots\ldots$

1. Since

$$S_1 = \tfrac{2}{3}\Theta,$$

we need, in order to obtain S_2, the value of $D(\Theta)$. According to the above rule

$$D(\Theta_x^2) = (\phi, N)_2 + (\psi, M)_2;$$

but

$$\begin{aligned}(M, \psi)_2 &= 3(\Phi\phi)(\phi\psi)^2\Phi_x\psi_x + 2(\Phi\phi)^2(\phi\psi)\psi_x^2,\\ (N, \phi)_2 &= -3(\Phi\psi)(\psi\phi)^2\Phi_x\phi_x - 2(\Phi\psi)^2(\psi\phi)\phi_x^2,\end{aligned}$$

therefore

$$D(\Theta_x^2) = -3J\Phi_x^2 + 2(\phi\psi)[(\Phi\phi)^2\psi_x^2 + (\Phi\psi)^2\phi_x^2].$$

But since*

$$(\vartheta, \phi)_2 = \tfrac{1}{6}J\phi, \quad (\vartheta, \phi)_3 = -\tfrac{3}{4}p,$$

we have

$$\left.\begin{aligned}(\Phi\phi)^2\phi_x &= 2J\phi_u^2\phi_x - 3p_u(ux),\\ (\Phi\psi)^2\psi_x &= 2J\psi_x^2\psi_u + 3\pi_u(ux),\end{aligned}\right\} \tag{18}$$

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* Caporali, Rend. della R. Acc. di Napoli, Marzo, 1883, §8.

hence

$$(\phi\psi)(\Phi\phi)^2\psi_x^2 = 2J(\phi\psi)\,\phi_u^2\psi_x^2 + 3p_u\,.\,\psi_u\psi_x^2,$$
$$(\phi\psi)(\Phi\psi)^2\phi_x^2 = 2J(\phi\psi)\,\psi_u^2\phi_x^2 + 3\pi_u\,.\,\phi_u\phi_x^2.$$

Collect the terms, apply Clebsch-Gordan's expansion and make use of the formulæ*

$$\Gamma = \tfrac{1}{2}(\phi\pi + \psi p) = J\vartheta + \Delta\nabla - \Theta^2, \tag{19}$$
$$2\Gamma_x^3\Gamma_u = \phi_x^2\phi_u\,.\,\pi_x + \psi_x^2\psi_u\,.\,p_x, \tag{19a}$$
$$2\Gamma_x^2\Gamma_u^2 = \phi_u\phi_x^2\,.\,\pi_u + \psi_u\psi_x^2\,.\,p_u, \tag{19b}$$

we obtain

$$D(\Theta_x^2) = 12\Gamma_u^2\Gamma_x^2 - 10J\vartheta_u^2\vartheta_x^2 - \tfrac{1}{3}J^2(ux)^2, \tag{20}$$

and putting $x = u$:

$$D(\Theta) = 12\Delta\nabla - 12\Theta^2 + 2J\vartheta. \tag{21}$$

The reduction formula (C) furnishes now for S_2 the value

$$S_2 = 18\Delta\nabla + 6J\vartheta - \tfrac{207}{25}\Theta^2. \tag{22}$$

2. To obtain S_3 we need $D(J)$, $D(\vartheta)$ and $D(\Delta\nabla)$.

$$\text{a).}\quad D(J) = (\phi,\,N)_3 + (M,\,\psi)_3$$
$$(M,\,\psi)_3 = 3(\Phi\phi)(\Phi\psi)(\phi\psi)^2 = 3(\Theta,\,\Phi)_2.$$

But†

$$(\Theta,\,\Phi)_2 = 6(\vartheta\Theta)^2\vartheta_u^2 + J\Theta_u^2 = -3\nu - J\Theta$$

where

$$\nu = (\nabla,\,\Delta)_1.$$

Hence

$$(M,\,\psi)_3 = -9\nu - 3J\Theta$$
$$(N,\,\phi)_3 = 9\nu + 3J\Theta,$$

consequently

$$D(J) = -18\nu - 6J\Theta. \tag{23}$$

$$\text{b).}\quad D(\vartheta_x^4) = (\phi,\,N)_1 + (M,\,\psi)_1$$
$$(M,\,\psi)_1 = 3(\Phi\phi)(\Phi\psi)\,\phi_x^2\psi_x^2 - 2(\Phi\phi)^2\,\phi_x\,.\,\psi_x^3$$
$$= -\tfrac{1}{2}(\Phi\phi)^2\,\phi_x\,.\,\psi_x^3 + \tfrac{3}{2}(\Phi\psi)^2\,\psi_x\,.\,\phi_x^3 - \tfrac{3}{2}(\phi\psi)^2\,\phi_x\psi_x\,.\,\Phi_x^2.$$

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* v. Gall, "Syzyganten cubischer Formen." Math. Annalen, Bd. 31, p. 435, and Gordan, "Invariantentheorie," II, p. 335.

† Caporali, l. c.

Hence putting $x = u$ and remembering that $(N, \phi)_1$ can be derived from $(M, \psi)_1$ by interchanging ϕ and ψ:

$$D(\vartheta) = 4J.\phi\psi - 18\Theta\vartheta. \tag{24}$$

c). $D(\Delta\nabla) = \Delta D(\nabla) + \nabla D(\Delta)$,

$$\begin{aligned} D(\Delta_x^2) &= 2(\phi, M)_2 = 6(\phi\phi')^2(\Phi\phi)\Phi_x\phi'_x + 4(\phi\phi')(\Phi\phi)^2\phi'^2_x \\ &= 2(\phi\phi')^2(\Phi\phi)\Phi_x\phi'_x = 2(\Phi, \Delta)_1. \end{aligned}$$

Similarly,

$$D(\nabla_x^2) = -2(\Phi, \nabla)_1,$$

hence

$$\begin{aligned} D(\Delta_x^2\nabla_x^2) &= -2[\Delta_x^2(\Phi, \nabla)_1 - \nabla_x^2(\Phi, \Delta)_1] \\ &= -2(\Delta\nabla)\Delta_x\nabla_x.\Phi_x^2. \end{aligned} \tag{25}$$

Hence, putting $x = u$,

$$D(\Delta\nabla) = 12\nu\vartheta. \tag{26}$$

The reduction formula (C) furnishes now for S_3 the value:

$$S_3 = \Theta\left[54\Delta\nabla - 54J\vartheta + \tfrac{81.27}{16}\Theta^2\right] + 24J^2\phi\psi + 108\nu\vartheta, \tag{27}$$

which agrees with Wiltheiss' result (Math. Annalen, 29, p. 297) if we make use of the syzygies (233), (323), (226), (224), (336), given by v. Gall in the above-named paper.

3. For the computation of S_4 we need $D(\phi\psi)$ and $D(\nu)$.

a). Since

$$D(\phi_x^3) = M_x^3, \quad D(\psi_x^3) = N_x^3,$$

we have

$$D(\phi_x^3\psi_x^3) = -3\Phi_x^2.\vartheta_x^4.$$

Hence for $x = u$:

$$D(\phi\psi) = -18\vartheta^2. \tag{28}$$

b). $D(\nu_x^2) = (\nabla_x^2, D(\Delta_x^2))_1 + (\Delta_x^2, D(\nabla_x^2))_1 = 2(\Delta\nabla)^2.\Phi_x^2 - (\Phi\nabla)^2.\Delta_x^2 - (\Phi\Delta)^2.\nabla_x^2.$

And making use of the relations*

$$\begin{aligned} (\vartheta, \Delta)_2 &= \quad (\Delta, \Theta)_1 - \tfrac{1}{3}J\Delta, \\ (\vartheta, \nabla)_2 &= -(\nabla, \Theta)_1 - \tfrac{1}{3}J\nabla, \end{aligned}$$

we obtain for $x = u$, always in Clebsch's notation,

$$D(\nu) = 6\nu\Theta + 12T\vartheta + 2J\Delta\nabla. \tag{29}$$

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* Caporali, l. c.

The result of the computation of S_4 is

$$S_4 = -\frac{3^6 . 11}{5^3}\Theta^4 + \frac{2^3 . 3^4 . 23}{5}\Theta^2 . \Delta\nabla - 2^3 . 3^4 . (\Delta\nabla)^2 + 2^3 . 3^4 . \Theta^2 . J\vartheta - 2^2 . 3^4 . J^2\vartheta^2$$
$$+ \frac{2^4 . 3^4 . 11}{5}\Theta . \vartheta\nu + \frac{2^5 . 3^2}{5}J^2 . \Theta . \phi\psi - 2^4 . 3^3 J\nu . \phi\psi + 2^4 . 3^4 \mathrm{T}\vartheta^2. \quad (30)$$

4. For the computation of S_5 the value of $D(\mathrm{T})$ is required.

$$\mathrm{T} = (\Delta, \nabla)_2,$$

hence

$$D(\mathrm{T}) = (\Delta_x^2, D(\nabla_x^2))_2 + (D(\Delta_x^2), \nabla_x^2)_2$$
$$= 4(\Phi\Delta)(\Phi\nabla)(\Delta\nabla)$$
$$= 24(\vartheta\Delta)(\vartheta\nabla)(\Delta\nabla)\vartheta_u^2 - 4J\nu_u^2.$$

But* $$(\vartheta\Delta)(\vartheta\nabla)(\Delta\nabla)\vartheta_u^2 - (\vartheta, \nu)_2 = -\tfrac{1}{2}p\pi - \tfrac{1}{6}J\nu,$$

hence $$D(\mathrm{T}) = -8J\nu - 12p\pi. \quad (31)$$

5. For the computation of S_6 the value of $D(p\pi)$ is required; this can be reduced to $D(\Delta\nabla, \Theta)_2$ as follows:

If we denote $$\Delta_x^2\nabla_x^2 = r_x^4,$$
we have from (19):

$$r_x^2 r_y^2 = \Gamma_x^2\Gamma_y^2 - J\vartheta_x^2\vartheta_y^2 + \Theta_x^2\Theta_y'^2 - \tfrac{1}{3}(\Theta\Theta')^2(xy)^2. \quad (32)$$

Hence

$$(\Delta\nabla, \Theta)_2 = (\Gamma, \Theta)_2 - J(\vartheta, \Theta)_2 + \tfrac{2}{3}(\Theta\Theta')^2 . \Theta.$$

But from (19b) follows:

$$(\Gamma, \Theta)_2 = -\tfrac{1}{2}p\pi;$$

further,

$$(\vartheta, \Theta)_2 = -\tfrac{1}{2}\nu - \tfrac{1}{3}J\Theta,$$
$$(\Theta\Theta')^2 = \mathrm{T} - \tfrac{1}{2}J^2,$$

hence $$2(\Delta\nabla, \Theta)_2 = -p\pi + J\nu + \tfrac{4}{3}\mathrm{T}\Theta.$$

Now $$D(\Delta_x^2\nabla_x^2, \Theta_x^2)_2 = (\Delta_x^2\nabla_x^2, D(\Theta_x^2))_2 + (D(\Delta_x^2\nabla_x^2), \Theta_x^2)_2.$$

From (20) follows:

$$(\Delta_x^2\nabla_x^2, D(\Theta_x^2))_2 = 12(\Gamma r)^2\Gamma_u^2 r_x^2 - 10J(\vartheta r)^2\vartheta_u^2 r_x^2 - \tfrac{1}{3}J^2 r_u^2 r_x^2.$$

Put $x = u$, make use of (32), and observe that†

$$(\Gamma, \Gamma)_2 = \Omega\vartheta, \ (\vartheta, \Gamma)_2 = \tfrac{1}{8}J\Gamma.$$

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* Càporali, l. c.

† Berzolari, Rendiconto dell Acc. di Napoli, Serie 2, vol. V, p. 77. Ω is used with the same sign as in Berzolari's and Càporali's papers, viz. $\Omega = -\frac{(p, \pi)_1}{2}$.

Thus we obtain

$$(\Delta\nabla,\ \Gamma)_2 = \Omega\vartheta - \tfrac{1}{2}\Theta\,.\,p\pi - \tfrac{1}{3}\mathrm{T}\Gamma;$$

further,*

$$(\Delta\nabla,\ \vartheta)_2 = -\tfrac{1}{3}J\Delta\nabla - \tfrac{1}{2}\Theta\nu - \tfrac{1}{3}\mathrm{T}\vartheta,$$

hence

$$(\Delta_x^2\nabla_x^2,\ D(\Theta_x^2))_2^{x=u} = 12\Omega\vartheta - 6\Theta p\pi + 5J\nu\,.\,\Theta + \tfrac{10}{3}\mathrm{T}J\vartheta - 4\mathrm{T}\Gamma + 3J^2\Delta\nabla.$$

On the other hand,

$$\big(D(\Delta_x^2\nabla_x^2),\ \Theta_x^2\big)_2 = 2(\nu_x^2\Phi_x^2,\ \Theta_x^2)_2 = \Phi_x^2(\nu,\ \Theta)_2 + \nu_x^2(\Phi,\ \Theta)_2 - \tfrac{2}{3}(\Phi\nu)^2.\,\Theta_x^2.$$

Observe that by definition

$$(\nu,\ \Theta)_2 = \Omega;$$

further

$$(\Phi,\ \Theta)_2 = -3\nu - J\Theta,\quad (\Phi,\ \nu)_2 = 3p\pi + 2J\nu.$$

Putting $x = u$ we have therefore

$$\big(D(\Delta_x^2\nabla_x^2),\ \Theta_x^2\big)_2^{x=u} = 6\Omega\vartheta - 3\nu^2 - \tfrac{4}{3}J\Theta\nu - 2\Theta p\pi.$$

And if we make use of the relation†

$$2\nu^2 = -4\Omega\vartheta + 2\Delta\nabla(2\mathrm{T} - J^2) - 4\mathrm{T}\Theta^2 + 4\Theta p\pi - J\nu\Theta$$

we finally obtain

$$D(p\pi) = -24\Theta\,.\,p\pi - 12\Omega\vartheta + 8J^2.\,\Delta\nabla + 8J\nu\Theta + 16\,\mathrm{T}J\vartheta. \qquad (34)$$

6. For the computation of S_7 the value of $D(\Omega)$ is required. Since

$$\Omega = (\Theta,\ \nu)_2,$$

we have

$$D(\Omega) = (\Theta_x^2,\ D(\nu_x^2))_2 + (D(\Theta_x^2),\ \nu_x^2)_2.$$

The value previously obtained for $D(\nu_x^2)$ can easily be transformed into

$$D(\nu_x^2) = 2J\Gamma_x^2\Gamma_u^2 - 2J^2\vartheta_u^2\vartheta_x^2 + 2J\Theta_u^2\Theta_x'^2 + 12\mathrm{T}\vartheta_u^2\vartheta_x^2$$
$$+ 6\nu_u^2\Theta_x^2 - (6\Omega - \tfrac{1}{3}J^3 - 2J\mathrm{T})(ux)^2$$

by means of the identity

$$\Delta_u^2\nabla_x^2 + \nabla_u^2\Delta_x^2 = 2\Gamma_u^2\Gamma_x^2 - 2J\vartheta_u^2\vartheta_x^2 + 2\Theta_u^2\Theta_x'^2 + \tfrac{1}{3}J^2(ux)^2.$$

Hence follows, by using previous results,

$$(\Theta_x^2,\ D(\nu_x^2))_2 = -Jp\pi - 2\nu J^2 - 6\Omega\Theta.$$

On the other hand we have, on account of (20),

$$(D(\Theta_x^2),\ \nu_x^2)_2 = 12(\Gamma\nu)^2\Gamma_u^2 - 10J(\vartheta\nu)^2\vartheta_u^2 - \tfrac{1}{3}J^2\nu_u^2.$$

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* Berzolari, l. c., p. 78.

† Cf. v. Gall, l. c., the syzygies $(444)_2$, and A) on p. 426, together with Berzolari, l. c., p. 74, (4).

But
$$(\vartheta,\, \nu)_2 = \tfrac{1}{2}\, p\pi + \tfrac{1}{6}\, J\nu,$$
$$(H_\vartheta,\, \nu)_2 = -\tfrac{1}{36}\, J^2\nu + \tfrac{1}{6}\, Jp\pi - \tfrac{1}{2}\, \Omega\Theta;$$

hence since: $3\Gamma = 6H_\vartheta + J\vartheta$,

$$(\Gamma,\, \nu)_2 = \tfrac{1}{2}\, Jp\pi - \Omega\Theta.$$

Thus we obtain
$$D\,(\Theta_x^2,\, \nu_x^2)_2 = Jp\pi - 2J^2\nu - 12\Omega\Theta$$
and
$$D\,(\Omega) = -4J^2\nu - 18\Omega\Theta. \tag{35}$$

Since $D\,(\Omega)$ *contains no new simultaneous concomitants of* ϕ *and* ψ, it follows that the terms following upon S_6 contain no other concomitants but those which have already made their appearance in the first six terms, and we have thus proved Wiltheiss-Brioschi's result:

Theorem V.

The successive terms in the expansion

$$\sigma_{\phi\psi}(u_1,\, u_2) = 1 + \frac{S_1}{2\,!} + \frac{S_2}{4\,!} + \frac{S_3}{6\,!} + \dots.$$

are expressible as integral functions of the following nine simultaneous concomitants of $\phi\,(u)$ *and* $\psi\,(u)$:

$$\begin{array}{llll} \phi\psi, & \vartheta = (\phi,\, \psi)_1, & \Theta = (\phi,\, \psi)_2, & J = (\phi,\, \psi)_3, \\ \Delta\nabla, & \nu = (\nabla,\, \Delta)_1, & & \mathrm{T} = (\Delta,\, \nabla)_2, \\ p\pi, & \Omega = -\dfrac{(p,\, \pi)_1}{2} & & \end{array}$$

and the effect of the operator D upon these forms is exhibited in the following table:

$$\begin{aligned} D\,(\Theta) &= 12\Delta\nabla - 12\Theta^2 + 2J\vartheta, \\ D\,(J) &= -18\nu - 6J\Theta, \\ D\,(\vartheta) &= 4J\,.\,\phi\psi - 18\Theta\vartheta, \\ D\,(\Delta\nabla) &= 12\nu\vartheta, \\ D\,(\phi\psi) &= -18\vartheta^2, \\ D\,(\nu) &= 6\nu\Theta + 12\mathrm{T}\vartheta + 2J\Delta\nabla, \\ D\,(\mathrm{T}) &= -8J\nu - 12p\pi, \\ D\,(p\pi) &= -24\Theta\,.\,p\pi - 12\Omega\vartheta + 8J^2\,.\,\Delta\nabla + 8J\nu\,.\,\Theta + 16\mathrm{T}J\vartheta, \\ D\,(\Omega) &= -4J^2\nu - 18\Omega\Theta. \end{aligned}$$

The above results can be compared with Wiltheiss' results (Math. Ann., Bd. 36, p. 153) as follows:

We obtain easily

$$(\Phi\phi)\,\Phi_x\phi_x^2 = 3\Theta_u^2\cdot\phi_x^3 - 12\Theta_u\Theta_x\cdot\phi_u\phi_x^2 + 6\Theta_x^2\cdot\phi_u^2\phi_x + 3\Delta_u^2\cdot\psi_x^3 + 2J\phi_u\phi_x^2\,(xu).$$

Hence it follows that Brioschi's operator D is expressible in terms of Wiltheiss' operators δ_1, δ_2, δ_3, δ_4, δ_5 as follows:

$$D = 3\,[3\delta_1 - 12\delta_2 + 6\delta_3 + 3\delta_4 + 2\delta_5]_{x=u}.$$

Our results agree exactly with Wiltheiss' results, whereas the values for $D(J)$, $D(T)$, $D(p\pi)$, $D(\Omega)$ do not agree with Brioschi's results.

UNIVERSITY OF CHICAGO, *October* 11*th*, 1898.

Note to Professor Craig's Memoir, "Displacements Depending on One, Two and Three Parameters in a Space of Four Dimensions."

BY E. JAHNKE.

In volume XX of this Journal Professor Craig, generalizing the kinematical quantities p, q, r of ordinary space, has introduced six quantities $p_{ij}(i, j = 1, 2, 3, 4)$, as follows:

$$\begin{array}{lll} p_{12} = -\sum \alpha_i \dfrac{d\beta_i}{dt}, & p_{13} = -\sum \gamma_i \dfrac{d\alpha_i}{dt}, & p_{23} = -\sum \beta_i \dfrac{d\gamma_i}{dt}, \\ p_{14} = -\sum \alpha_i \dfrac{d\delta_i}{dt}, & p_{24} = -\sum \beta_i \dfrac{d\delta_i}{dt}, & p_{34} = -\sum \gamma_i \dfrac{d\delta_i}{dt}, \end{array} \quad (i = 1, 2, 3, 4)$$

and established the following system of six differential equations analogous to the kinematical differential equations in the space of three dimensions:

$$\begin{aligned} \frac{d\alpha}{dt} &= \quad p_{12}\beta - p_{13}\gamma + p_{14}\delta, \\ \frac{d\beta}{dt} &= -p_{12}\alpha + p_{23}\gamma + p_{24}\delta, \\ \frac{d\gamma}{dt} &= \quad p_{13}\alpha - p_{23}\beta + p_{34}\delta, \\ \frac{d\delta}{dt} &= -p_{14}\alpha - p_{24}\beta - p_{34}\gamma. \end{aligned}$$

My memoir, "Ueber einen Zusammenhang zwischen den Elementen orthogonaler Neuner- und Sechzehnersysteme," Crelle's T., Bd. CXVIII, 224–233 appeared before. I have introduced there not only the six differential quantities

$$g\,p_{rs} = -(g_{1i}\,dg_{1j} + g_{2i}\,dg_{2j} + g_{3i}\,dg_{3j} + g_{4i}\,dg_{4j}),$$

but six other differential quantities

$$g\,v_{rs} = g_{i1}\,dg_{j1} + g_{i2}\,dg_{j2} + g_{i3}\,dg_{j3} + g_{i4}\,dg_{j4}, \quad i, j, r, s = 1, 4, 2, 3$$
$$= 2, 4, 3, 1$$
$$= 3, 4, 1, 2$$

where
$$g = g_{1i}^2 + g_{2i}^2 + g_{3i}^2 + g_{4i}^2 = g_{i1}^2 + g_{i2}^2 + g_{i3}^2 + g_{i4}^2.$$

I have obtained, with the aid of them, *two* systems of differential identities as follows:

$$g_{1i}\,d\log\frac{g_{1i}}{\sqrt{g}} = g_{1i}p_{rs} + g_{1r}p_{sj} + g_{1s}p_{jr} = -g_{2i}v_{34} - g_{3i}v_{42} - g_{4i}v_{23},$$

$$g_{2i}\,d\log\frac{g_{2i}}{\sqrt{g}} = g_{2j}p_{rs} + g_{2r}p_{sj} + g_{2s}p_{jr} = -g_{3i}v_{14} - g_{4i}v_{31} - g_{1i}v_{43},$$

$$g_{3i}\,d\log\frac{g_{3i}}{\sqrt{g}} = g_{3j}p_{r} + g_{3r}p_{sj} + g_{3s}p_{jr} = -g_{4i}v_{12} - g_{1i}v_{24} - g_{2i}v_{41},$$

$$g_{4i}\,d\log\frac{g_{4i}}{\sqrt{g}} = g_{4j}p_{rs} + g_{4r}p_{sj} + g_{4s}p_{jr} = -g_{1i}v_{32} - g_{2i}v_{13} - g_{3i}v_{21}.^{*}$$

Professor Craig's system is, with a little difference in notation, a special case of the first of my systems; it is obtained when we suppose $g = 1$.

I have employed my systems, it is true, for another purpose than Professor Craig's. I have deduced from them the general form which may be given to the differential equations of all problems referring to rotation. It is to be remarked that the introduction of the six differential quantities v_{rs} is necessary for the *complete* solution of dynamical problems. (Compare C. R. CXXVI, 1014 and Crelle's T., Bd. CXIX, 240.)

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[*] In my memoir is an erratum. Instead of dg_{mn} there must be read $g_{mn}\,d\log\frac{g_{mn}}{\sqrt{g}}$.

Determination of the Structure of all Linear Homogeneous Groups in a Galois Field which are Defined by a Quadratic Invariant.

By Leonard Eugene Dickson.

Following the study of certain classes of finite linear groups defined by a quadratic invariant, it seems desirable to have a complete determination of this important type of groups. Besides the work of Jordan[*] on the two hypoabelian groups in the field of integers taken modulo 2, and the writer's generalization[†] of the first hypoabelian group to the Galois field of order 2^n, the structures of the orthogonal group[‡] on m indices in the Galois field of order p^n (aside from certain low values of m, n, p) and of the group[||] in the same field, leaving invariant the quadratic form $\sum_{i=1}^{m}\xi_i\eta_i$, have been previously determined by the writer.

By setting up a complete set of canonical forms for quadratic forms in m variables in every Galois field, we are able to prove that there exist but two new distinct types of groups defined by a quadratic invariant, one of these being a generalization of the second hypoabelian group of Jordan. Two new systems of simple groups are thus obtained [see §56]. The investigation completes and correlates the results of the earlier papers. It has been the aim throughout to devise methods which require as few separations into cases and special treatments of lower cases as possible. The earlier methods for the orthogonal group have been abandoned in the main.

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[*] Traité des Substitutions, pp. 195–218 and p. 440.

[†] "On the First Hypoabelian Group Generalized," The Quarterly Journal, pp. 1–16, 1898; "The Structure of the Hypoabelian Groups," Bulletin of the American Mathematical Society, pp. 495–510 July, 1898.

[‡] "Systems of Simple Groups derived from the Orthogonal Group," Proceedings of the California Academy of Sciences, vol. I, No. 4, 1898, and No. 5, 1899; also Bulletin of the Amer. Math. Society, Feb., 1898, and May, 1898.

[||] "The Structure of Certain Linear Groups with Quadratic Invariants," Proceedings of the London Mathematical Society, vol. XXX, pp. 70–98.

1. Consider a quadratic function ϕ homogeneous in m variables $\xi_1, \xi_2, \ldots, \xi_m$ and having as coefficients marks* of the Galois field of order p^n. We restrict ourselves to forms ϕ of determinant not zero in the $GF[p^n]$ and suppose, for the present, that $p > 2$. By an investigation analogous to that in Bachmann, Zahlentheorie, IV, pp. 409–412, we can prove that there exists a linear homogeneous substitution T on the variables $\xi_1, \ldots, \xi_m$ with coefficients belonging to the $GF[p^n]$ which transforms ϕ into

$$f_s \equiv \sum_{i=1}^{s} \xi_i^2 + \nu \sum_{i=s+1}^{m} \xi_i^2,$$

ν denoting any particular not-square in the $GF[p^n]$. Further, we can transform f_s into f_{s+2}. Consider indeed the substitution of determinant $\alpha^2 + \beta^2$,

$$\xi_i' = \alpha\xi_i - \beta\xi_j, \quad \xi_j' = \beta\xi_i + \alpha\xi_j.$$

It transforms $\xi_i^2 + \xi_j^2$ into $(\alpha^2 + \beta^2)(\xi_i^2 + \xi_j^2)$. By the theorem quoted in §3, there exist marks α, β in the $GF[p^n]$, $p > 2$, for which $\alpha^2 + \beta^2 = \nu$, a not-square. Hence in the form f_s we can replace $\xi_i^2 + \xi_j^2$ by $\nu\xi_i^2 + \nu\xi_j^2$ and inversely. We have therefore two canonical forms, f_m and f_{m-1}.

For m odd, the form f_{m-1} can be transformed into

$$f_0 \equiv \nu(\xi_1^2 + \xi_2^2 + \cdots + \xi_m^2).$$

But the group leaving f_0 invariant leaves also $f_m \equiv \xi_1^2 + \cdots + \xi_m^2$ invariant. We may therefore state the result:

Theorem: *Every linear homogeneous group in the* $GF[p^n]$, $p > 2$, *defined by a quadratic invariant of determinant not zero, can be transformed by a linear homogeneous substitution belonging to the field into one of the two groups:*

1°. *The orthogonal group, with the invariant* $\sum_{i=1}^{m} \xi_i^2$.

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*The theory of Galois is used in its abstract form, as presented by Moore in the Congress Mathematical Papers, 1898.

2°. *The group on an even number of indices with the invariant* $\sum_{i=1}^{m-1}\xi_i^2 + \nu\xi_m^2$.

2. Denote by $G^{(s)}_{m,\,p^n}$ the group leaving f_s invariant. The conditions that any substitution

$$S:\quad \xi_i' = \sum_{j=1}^{m}\alpha_{ij}\xi_j \qquad (i = 1, 2, \ldots, m)$$

shall leave f_s invariant are as follows:*

$$(1).\quad \alpha_{1j}^2 + \alpha_{2j}^2 + \ldots + \alpha_{sj}^2 + \nu\,(\alpha_{s+1j}^2 + \ldots + \alpha_{mj}^2) = \begin{cases} 1, & (j \leqq s) \\ \nu, & (j > s) \end{cases}$$

$$(2).\quad \alpha_{1j}\alpha_{1k} + \ldots + \alpha_{sj}\alpha_{sk} + \nu\,(\alpha_{s+1j}\alpha_{s+1k} + \ldots + \alpha_{mj}\alpha_{mk}) = 0.$$

$$(j, k = 1, \ldots, m;\ j \neq k)$$

It follows that the reciprocal of S is

$$S^{-1}:\ \begin{cases} \xi_i' = \displaystyle\sum_{j=1}^{m}\alpha_{ji}\xi_j + \nu\sum_{j=s+1}^{m}\alpha_{ji}\xi_j, & (i = 1, \ldots, s) \\ \xi_i' = \dfrac{1}{\nu}\displaystyle\sum_{j=1}^{s}\alpha_{ji}\xi_j + \sum_{j=s+1}^{m}\alpha_{ji}\xi_j. & (i = s+1, \ldots, m) \end{cases}$$

The determinant of S^{-1} is seen to be equal to the determinant Δ of S. Hence $\Delta^2 = 1$, being the determinant of $S^{-1}S \equiv 1$. Writing the relations (1) and (2) for the substitution S^{-1}, we obtain the relations

$$(1').\quad \alpha_{j1}^2 + \alpha_{j2}^2 + \ldots + \alpha_{js}^2 + \frac{1}{\nu}(\alpha_{js+1}^2 + \ldots + \alpha_{jm}^2) = \begin{cases} 1, & (j \leqq s) \\ 1/\nu, & (j > s) \end{cases}$$

$$(2').\quad \alpha_{j1}\alpha_{k1} + \ldots + \alpha_{js}\alpha_{ks} + \frac{1}{\nu}(\alpha_{js+1}\alpha_{ks+1} + \ldots + \alpha_{jm}\alpha_{km}) = 0.$$

$$(j, k = 1, \ldots, m;\ j \neq k)$$

These relations are together equivalent to the set (1), (2).

3. Lemma: *The number of systems of solutions* $\xi_1, \ldots, \xi_{2m}$ *in the* $GF[p^n]$, $p > 2$, *of the equation*

$$\alpha_1\xi_1^2 + \alpha_2\xi_2^2 + \ldots + \alpha_{2m}\xi_{2m}^2 = \kappa,$$

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* The conditions (2) do not occur if $p = 2$, a case now excluded.

where every α_j *is a mark* $\neq 0$ *of the field, is*

$$p^{n(2m-1)} - \nu p^{n(m-1)}, \qquad (\varkappa \neq 0)$$
$$p^{n(2m-1)} + \nu\left(p^{nm} - p^{n(m-1)}\right), \qquad (\varkappa = 0)$$

where ν *is* $+1$ *or* -1 *according as* $(-1)^m\alpha_1\alpha_2\ldots\alpha_{2m}$ *is a square or a not-square in the field. The number of solutions of*

$$\alpha_1\xi_1^2 + \alpha_2\xi_2^2 + \ldots + \alpha_{2m+1}\xi_{2m+1}^2 = \varkappa$$

is $p^{2nm} + \nu' p^{nm}$, *where* ν' *is* $+1$, -1 *or* 0 *according as* $(-1)^m\alpha_1\alpha_2\ldots\alpha_{2m+1}\varkappa$ *is a square, not-square or zero in the* $GF[p^n]$.

These results follow from an immediate generalization of §§197–199, 201–212 of Jordan, "Traité des Substitutions," or of pp. 486–491 of Bachmann, Zahlentheorie, IV.

4. Lemma: *If* S *denote the number of squares* in the* $GF[p^n]$ *followed by squares and* N *the number of squares followed by not-squares, we have*

$$S = \tfrac{1}{4}(p^n - 5), \quad N = \tfrac{1}{4}(p^n - 1), \quad \textit{if } -1 = \textit{square};$$
$$S = \tfrac{1}{4}(p^n - 3), \quad N = \tfrac{1}{4}(p^n + 1), \quad \textit{if } -1 = \textit{not-square}.$$

Indeed, the number of sets of solutions ξ, η in the $GF[p^n]$ of the equation

$$\eta^2 = \xi^2 + 1$$

is always $p^n - 1$ (by §3). These solutions are of three kinds:

$$1^\circ.\quad \xi = 0 \;, \quad \eta = \pm 1;$$
$$2^\circ.\quad \xi^2 = -1, \quad \eta = 0 \;,$$

occurring when -1 is a square;

$$3^\circ.\quad \xi^2 = \alpha \neq 0, \quad \eta^2 = \alpha + 1 \neq 0,$$

giving $4S$ sets of solutions ξ, η.

Hence, if -1 be a square, we have

$$p^n - 1 = 2 + 2 + 4S, \quad N + S + 1 = \tfrac{1}{2}(p^n - 1).$$

If -1 be a not-square, we have

$$p^n - 1 = 2 + 4S, \quad N + S = \tfrac{1}{2}(p^n - 1).$$

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* The mark zero is not reckoned as a square.

5. Theorem: *The order of the group* $G^{(s)}_{m,\,p^n}$ *is, for m odd,*

$$2\,(p^{n(m-1)}-1)\,p^{n(m-2)}\,(p^{n(m-3)}-1)\,p^{n(m-4)}\,\ldots\ldots\,(p^{2n}-1)\,p^n,$$

*and, for m even,**

$$2\left[p^{n(m-1)}-(-1)^s\varepsilon^{\frac{m}{2}}p^{n\left(\frac{m}{2}-1\right)}\right](p^{n(m-2)}-1)\,p^{n(m-3)}\,\ldots\ldots\,(p^{2n}-1)\,p^n,$$

where $\varepsilon=\pm 1$ *according as* p^n *is of the form* $4l\pm 1$.

Let $N^{(s)}_m$ denote the number of substitutions S, S', ... in the group which leave ξ_1 fixed. Let a general substitution T of the group replace ξ_1 by

$$F_1\equiv\sum_{j=1}^{m}\alpha_{1j}\xi_j,\quad \sum_{j=1}^{s}\alpha_{1j}^2+\frac{1}{\nu}\sum_{j=s+1}^{m}\alpha_{1j}^2=1.$$

The $N^{(s)}_m$ substitutions TS, TS', ..., and no others, will replace ξ_1 by F_1. If, therefore, $P^{(s)}_m$ denotes the number of distinct linear functions F_1 by which the substitutions of the group can replace ξ_1, we have for the order of the group,

$$\Omega^{(s)}_{m,\,p^n}=N^{(s)}_mP^{(s)}_m.$$

For the substitutions S, S', ..., we have

$$\alpha_{11}=1,\quad \alpha_{1j}=0.\qquad (j=2,\ldots\ldots,m)$$

Then by the relations (2′),

$$\alpha_{k1}=0.\qquad (k=2,3,\ldots\ldots,m)$$

The substitutions S, S', ..., therefore belong to the group $G^{(s-1)}_{m-1,\,p^n}$, leaving invariant

$$\sum_{i=2}^{s}\xi_i^2+\nu\sum_{i=s+1}^{m}\xi_i^2.$$

Hence

$$N^{(s)}_m=\Omega^{(s-1)}_{m-1,\,p^n}.$$

Repeating this argument, we find that

$$\Omega^{(s)}_{m,\,p^n}=P^{(s)}_m\Omega^{(s-1)}_{m-1,\,p^n}=P^{(s)}_mP^{(s-1)}_{m-1}\ldots\ldots P^{(2)}_{m-s+2}\Omega^{(1)}_{m-s+1,\,p^n},$$

where $\Omega^{(1)}_{m-s+1,\,p^n}$ is the order of the group leaving invariant ξ_m^2 or $\xi_{m-1}^2+\nu\xi_m^2$, according as $s=m$ or $s=m-1$, and therefore equals 2 or $2P^{(1)}_2$ respectively. Hence

$$\Omega^{(m)}_{m,\,p^n}=P^{(m)}_m\;P^{(m-1)}_{m-1}\ldots\ldots P^{(2)}_2\,.\,2,$$
$$\Omega^{(m-1)}_{m,\,p^n}=P^{(m-1)}_mP^{(m-2)}_{m-1}\ldots\ldots P^{(2)}_3P^{(1)}_2\,.\,2.$$

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*For $m=2$, the terms at the end of the formula do not occur.

It is proven in §§7–12 that the number $P_k^{(l)}$ is equal to the number of sets of solutions in the $GF[p^n]$ of the equation

$$\sum_{j=1}^{l} a_j^2 + \frac{1}{\nu} \sum_{j=l+1}^{k} a_j^2 = 1,$$

which, by §3, is seen to be as follows:

$$p^{n(k-1)} - (-1)^{k-l} \varepsilon^{\frac{k}{2}} p^{n\left(\frac{k}{2}-1\right)}, \qquad (k \text{ even})$$

$$p^{n(k-1)} + (-1)^{k-l} \varepsilon^{\frac{k-1}{2}} p^{n(k-1)/2}, \qquad (k \text{ odd})$$

ε denoting ± 1 according as -1 is a square or a not-square in the $GF[p^n]$. Whether t be even or odd, we have

$$P_{2t+1}^{(l)} \cdot P_{2t}^{(l-1)} = (p^{2nt} - 1)\, p^{n(2t-1)}.$$

We derive at once the expressions for the order $\Omega_{m,\, p^n}^{(s)}$ as given in the theorem.

6. Theorem: *The orthogonal group $G_{m,\, p^n}^{(m)}$ is generated by the substitutions [only the indices altered being written],*

$$O_i: \qquad \xi_i' = -\xi_i,$$

$$O_{i,j}^{\alpha,\beta}: \quad \begin{cases} \xi_i' = \alpha\xi_i + \beta\xi_j, \\ \xi_j' = -\beta\xi_i + \alpha\xi_j, \end{cases} \qquad (\alpha^2 + \beta^2 = 1)$$

*with the two following exceptions:**

for $p^n = 5$, $m \geqq 3$, we may take as the necessary additional generator the substitution of period two,

$$R: \begin{cases} \xi_1' = \xi_1 + \xi_2 + 2\xi_3, \\ \xi_2' = \xi_1 + 2\xi_2 + \xi_3, \\ \xi_3' = 2\xi_1 + \xi_2 + \xi_3; \end{cases}$$

for $p^n = 3$, $m \geqq 4$, we may choose as the additional generator

$$W: \begin{cases} \xi_1' = \xi_1 - \xi_2 - \xi_3 - \xi_4, \\ \xi_2' = \xi_1 - \xi_2 + \xi_3 + \xi_4, \\ \xi_3' = \xi_1 + \xi_2 - \xi_3 + \xi_4, \\ \xi_4' = \xi_1 + \xi_2 + \xi_3 - \xi_4. \end{cases} \qquad (W^2 = 1)$$

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* These exceptions were overlooked by Jordan in his treatment of the case $n = 1$.

The group $G_{m,p^n}^{(m-1)}$ *is generated by the substitutions* C_i, $O_{i,j}^{\alpha,\beta}$ $(i, j < m)$ *together with*

$$O_{i,m}^{\gamma,\delta}: \begin{cases} \xi_i' = \gamma\xi_i + \delta\xi_m, \\ \xi_m' = -\dfrac{\delta}{\nu}\xi_i + \gamma\xi_m, \end{cases} \qquad \left(\gamma^2 + \frac{1}{\nu}\delta^2 = 1\right)$$

an additional generator being necessary if $p^n = 3$, $m \geqq 3$, *viz.*

$$V_{1,2,m}: \begin{cases} \xi_1' = \xi_1 - \xi_2 - \xi_m, \\ \xi_2' = \xi_1 - \xi_2 + \xi_m, \\ \xi_m' = -\xi_1 - \xi_2 \end{cases} \qquad (V^3 = 1)$$

Our theorem is evident if $m = 2$. For $m \geqq 3$, it will follow from §5 by applying the results of §§7–12.

7. Theorem: *If* α_1, α_2, α_3 *be any set of solutions in the* $GF[p^n]$ *of the equation*

$$\alpha_1^2 + \alpha_2^2 + \frac{1}{\mu}\alpha_3^2 = 1$$

(*where* $\mu = 1$ *or the not-square* ν), *there exists a substitution* S *derived from the generators of* §6 *which leave invariant*

$$\xi_1^2 + \xi_2^2 + \mu\xi_3^2,$$

such that S *will replace* ξ_1 *by* $\alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3$.

The proposition follows at once if $1 - \alpha_1^2$ or $1 - \alpha_2^2$ be a square (excluding zero) in the $GF[p^n]$. For, if $1 - \alpha_2^2 = \tau^2$, then

$$\frac{\alpha_1^2}{\tau^2} + \frac{1}{\mu}\frac{\alpha_3^2}{\tau^2} = 1.$$

We may therefore take

$$S = (\xi_1\xi_2)\, O_{2,3}^{\frac{\alpha_1}{\tau}, -\frac{\alpha_3}{\tau}}\, O_{1,2}^{\alpha_2,\tau}.$$

The proposition is true for the quantities α_1, α_2, α_3 *if true for*

$$\alpha_1' \equiv \alpha_1, \quad \alpha_2' \equiv \beta\alpha_2 + \frac{\gamma}{\mu}\alpha_3, \quad \alpha_3' \equiv -\gamma\alpha_2 + \beta\alpha_3,$$

where

$$\beta^2 + \frac{1}{\mu}\gamma^2 = 1.$$

We notice that

$$\alpha_1'^2 + \alpha_2'^2 + \frac{1}{\mu}\alpha_3'^2 = \alpha_1^2 + \alpha_2^2 + \frac{1}{\mu}\alpha_3^2 = 1. \tag{3}$$

Then, if the group contains a substitution S' replacing ξ_1 by $\alpha_1'\xi_1 + \alpha_2'\xi_2 + \alpha_3'\xi_3$, it will contain the product $O_{2,3}^{\beta,\gamma} S'$ which replaces ξ_1 by $\alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3$.

Similarly, the proposition is true for $\alpha_1, \alpha_2, \alpha_3$ *if true for the quantities*

$$\alpha_1' \equiv \alpha_1\rho - \alpha_2\sigma, \quad \alpha_2' \equiv \alpha_1\sigma + \alpha_2\rho, \quad \alpha_3' \equiv \alpha_3,$$

where

$$\rho^2 + \sigma^2 = 1.$$

8. Consider first the case in which -1 is a not-square in the $GF[p^n]$. There are (by §3) $p^n + 1$ sets of solutions ρ, σ in the field of the equation $\rho^2 + \sigma^2 = 1$. Not more than two of these sets of solutions give the same value to

$$\alpha_2' \equiv \alpha_1\sigma + \alpha_2\rho.$$

Indeed, by eliminating σ, we obtain a quadratic for ρ. Hence α_2' takes at least $\frac{1}{2}(p^n + 1)$ distinct values. But by §4 there are exactly $\frac{1}{2}(p^n - 3)$ distinct marks $\eta \neq 0$ for which $\eta^2 - 1$ is a square, i. e. for which $1 - \eta^2$ is a not-square. Hence there exist at least two values of α_2' for which $1 - \alpha_2'^2$ is a square or zero. If it be a square, our theorem follows from the remark at the end of the last paragraph.

It remains to consider the case $\alpha_2'^2 = 1$. Then by (3),

$$\alpha_1'^2 = -\frac{1}{\mu}\alpha_3'^2.$$

If $\mu = 1$, we have $\alpha_1' = \alpha_3' = 0$ and the theorem is evident. If μ be a not-square, we may take $\mu = -1$. Then

$$\alpha_1' = \pm\alpha_3', \quad \alpha_2'^2 = 1.$$

As in §7, the theorem is true for α_1', α_2', α_3' if true for the quantities

$$\alpha_1'' \equiv \alpha_1'\beta - \alpha_3'\gamma, \quad \alpha_2'' \equiv \alpha_2', \quad \alpha_3'' \equiv -\gamma\alpha_1' + \beta\alpha_3',$$

where

$$\beta^2 - \gamma^2 = 1.$$

The p^n-1 solutions of this equation are given by

$$\beta = \tfrac{1}{2}\left(\tau + \frac{1}{\tau}\right), \quad \mp\gamma = \tfrac{1}{2}\left(\tau - \frac{1}{\tau}\right),$$

where τ runs through the marks $\neq 0$ of the $GF[p^n]$. Hence $\beta \mp \gamma$ may be given an arbitrary value $\tau \neq 0$ in the field. The theorem being evident if $\alpha_1' = 0$, we exclude this case. Then $\alpha_1'' \equiv \alpha_1'(\beta \mp \gamma)$ may be made to assume an arbitrary value except zero, and hence, if $p^n > 3$, a value for which $1 - \alpha_1''^2$ is a square in the field.

It remains to consider, when $p^n = 3$, the case in which

$$\alpha_1' = \pm \alpha_3' \neq 0, \quad \alpha_2'^2 = 1, \quad \mu = -1.$$

Since α_1', α_2', α_3' are each ± 1, we may evidently take

$$S = CV,$$

where C is a product formed from C_1, C_2, C_3.

9. Suppose next that -1 is the square of a mark I belonging to the $GF[p^n]$. If μ be a not-square, there exist p^n+1 sets of solutions in the field of the equation

$$\beta^2 + \frac{1}{\mu}\gamma^2 = 1. \tag{4}$$

By §7, the theorem is true if proven true for the values

$$\alpha_1' \equiv \alpha_1, \quad \alpha_2' \equiv \beta\alpha_2 + \frac{\gamma}{\mu}\alpha_3, \quad \alpha_3' \equiv -\gamma\alpha_2 + \beta\alpha_3.$$

There are at least $\frac{1}{2}(p^n+1)$ sets of solutions of (4) for which the values of α_2' are distinct; for upon eliminating β we obtain a quadratic for γ. But by §4 there exist only $\frac{1}{2}(p^n-1)$ marks $I\xi$, and hence as many distinct values of ξ, for which $(I\xi)^2 + 1 \equiv 1 - \xi^2$ is a not-square. Hence at least one set of solutions of (4) will make $1 - \alpha_2'^2$ a square or zero. If it be a square, the theorem follows from §7. If it be zero, (3) gives

$$\alpha_1'^2 = -\frac{1}{\mu}\alpha_3'^2.$$

Since μ is a not-square and -1 a square, we have

$$\alpha_1' = \alpha_3' = 0, \quad \alpha_2'^2 = 1,$$

so that we may take as the required substitution

$$\xi_1' = \alpha_2'\xi_2, \quad \xi_2' = \xi_1, \quad \xi_3' = \xi_3.$$

10. There remains the case in which -1 and μ are both squares. We may take $\mu = 1$, so that we have

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1.$$

There are now $p^n - 1$ sets of solutions of (4). These give at least $\frac{1}{2}(p^n - 1)$ distinct values of α_2'. Hence α_2' must take a value for which $1 - \alpha_2'^2$ is a square or zero or else be capable of taking *every* value for which $1 - \alpha_2'^2$ is a not-square. If it be a square, the theorem follows at once. If it be zero, we have

$$\alpha_2'^2 = 1, \quad \alpha_1'^2 + \alpha_3'^2 = 0. \tag{5}$$

If $\alpha_3' = 0$, the proposition follows at once. Suppose that $\alpha_3' \neq 0$. The proposition will be true for $\alpha_1', \alpha_2', \alpha_3'$ if proven for

$$\alpha_1'' \equiv \alpha_1'\rho - \alpha_3'\sigma, \quad \alpha_2'' \equiv \alpha_2' \quad \alpha_3'' \equiv \alpha_1'\sigma + \alpha_3'\rho,$$

where

$$\rho^2 + \sigma^2 = 1.$$

We can give to α_1'' an arbitrary value $\neq 0$ in the $GF[p^n]$. Indeed, on eliminating σ, we obtain for ρ the *linear* equation (the coefficient of ρ^2 being zero),

$$\rho^2\left(1 + \frac{\alpha_1'^2}{\alpha_3'^2}\right) - 2\frac{\alpha_1''\alpha_1'}{\alpha_3'^2}\rho + \left(\frac{\alpha_1''}{\alpha_3'}\right)^2 = 1.$$

But by §4 there are $\frac{1}{4}(p^n - 5)$ squares τ^2 for which $\tau^2 - 1$ and hence also $1 - \tau^2$ is a square. Our theorem therefore follows if $p^n \neq 5$.

There remains the case in which α_2' may take every one of the values for which $1 - \alpha_2'^2$ is a not-square. Repeating the same arguments for the quantities $\alpha_1'', \alpha_2'' \alpha_3''$, we find that, for $p^n \neq 5$, the only case in which the theorem is not proven is that in which α_1'' and α_2'' may each take every one of the $\frac{1}{2}(p^n - 1)$

values δ for which $1-\delta^2$ is a not-square. Hence if our theorem be true for one such set of quantities

$$\alpha_1''=\delta_1, \quad \alpha_2''=\delta_2, \quad \alpha_3'',$$

it is true for every set; if false for one, it is false for every set. Further, we have

$${\alpha_1''}^2+{\alpha_2''}^2+{\alpha_3''}^2={\alpha_1'}^2+{\alpha_2'}^2+{\alpha_3'}^2=\alpha_1^2+\alpha_2^2+\alpha_3^2=1.$$

Hence, whatever one of the $\{\frac{1}{2}(p^n-1)\}^2$ pairs of values we take for δ_1, δ_2, we can satisfy the equation

$$\delta_1^2+\delta_2^2+\delta_3^2=1$$

in two ways, viz. by $\delta_3=\pm\alpha_3''$. This equation has therefore $\frac{1}{2}(p^n-1)^2$ sets of solutions δ_1, δ_2, δ_3 for which $1-\delta_1^2$ and $1-\delta_2^2$ are not-squares. By virtue of the substitution C_3, the proposition is true for $\delta_1, \delta_2, -\delta_3$ if it be true for $\delta_1, \delta_2, +\delta_3$. If therefore our theorem be not always true, it will be false for all of the above $\frac{1}{2}(p^n-1)^2$ sets of values. It has been proven true for all other sets of solutions of

$$\alpha_1^2+\alpha_2^2+\alpha_3^2=1.$$

The total number of sets of solutions is (by §3) $p^{2n}+p^n$, -1 being a square.

The substitutions of the ternary orthogonal group would therefore replace ξ_1 by

$$R_3\equiv p^{2n}+p^n-\tfrac{1}{2}(p^n-1)^2=\tfrac{1}{2}(p^{2n}+4p^n-1)$$

distinct linear functions. The number of substitutions leaving ξ_1 fixed is clearly $2(p^n-1)$. The order of the group would thus be

$$(p^{2n}+4p^n-1)(p^n-1).$$

This number must divide the order of the general ternary linear homogeneous group in the $GF[p^n]$, viz.

$$(p^{3n}-1)(p^{3n}-p^n)(p^{3n}-p^{2n}).$$

Hence $p^{2n}+4p^n-1$, which is relatively prime to p, must divide $(p^{3n}-1)(p^{2n}-1)$ and hence also

$$4p^n(p^{3n}-1)\equiv 4p^n\{(p^n-4)(p^{2n}+4p^n-1)+17p^n-5\}.$$

It must therefore divide $4(17p^n - 5)$ and hence also

$$20(p^{2n} + 4p^n - 1) - (68p^n - 20) = p^n(20p^n + 12).$$

Hence $(p^n + 2)^2 - 5$ must divide 304; indeed

$$3(68p^n - 20) + 5(20p^n + 12) = 304p^n.$$

Hence $$p^n + 2 < 18 > \sqrt{309}.$$

But the only values of $p^n < 16$ for which -1 is a square in the $GF[p^n]$ are $p^n = 13, 9, 5$. For none of these is $(p^n + 2)^2 - 5$ a divisor of $304 \equiv 16.19$.

11. There remains the case $p^n = 5$, $\mu = 1$, not treated in §10 in the two following sub-cases:

For the case in which (5) holds, we have

$$\alpha_2'^2 = 1, \quad \alpha_1'^2 = \pm 1, \quad \alpha_3'^2 = \mp 1,$$

the only squares being ± 1. We may therefore take $S \doteq TR$, T being derived from C_1, C_2, C_3 and $(\xi_1\xi_3)$.

For the case in which $1 - \alpha_2'^2$ is a not-square, we have

$$\alpha_2'^2 = -1, \quad \alpha_1'^2 = 1, \quad \alpha_3'^2 = 1.$$

Then will $S = C(\xi_2\xi_3)R$, where C is derived from C_1, C_2, C_3, replace ξ_1 by $\alpha_1'\xi_1 + \alpha_2'\xi_2 + \alpha_3'\xi_3$.

Note: R cannot be derived from the C_i and $O_{i,j}^{\alpha,\beta}$; indeed, the latter are of the form C_iC_j, or the identity, or

$$\xi_i' = \pm \xi_j, \quad \xi_j' = \mp \xi_i.$$

12. Theorem: *If $\alpha_1, \alpha_2, \ldots, \alpha_m$ be any set of solutions in the $GF[p^n]$ of*

$$\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_{m-1}^2 + \frac{1}{\mu}\alpha_m^2 = 1,$$

there exists a substitution S derived from the generators of §6 which leave invariant $\sum_{i=1}^{m-1}\xi_i^2 + \mu\xi_m^2$ such that S will replace ξ_1 by $\sum_{j=1}^{m}\alpha_j\xi_j$.

The proposition being true for $m = 2$ and $m = 3$, we will make a proof by induction from $m - 1$ to m, supposing $m > 3$.

Consider first the cases in which every sum of three of the terms $\alpha_1^2, \alpha_2^2, \ldots\ldots, \alpha_{m-1}^2, \frac{1}{\mu}\alpha_m^2$ is zero. These terms must all be equal and therefore

$$m\alpha_1^2 = 1, \quad 3\alpha_1^2 = 0, \quad \mu = \text{square}.$$

Hence $p = 3$, while m is of the form $3k + 2$ or $3k + 1$.

If $m = 3k + 2$, we have $1 - \alpha_1^2 = \alpha_1^2 \neq 0$, so that the theorem is reduced by §7 to the case of $m - 1$ indices.

If $m = 3k + 1$, we must have $\alpha_1^2 = 1$. But the product $O_{1,2}^{\alpha,\beta} S$ will replace ξ_1 by $\alpha_1'\xi_1 + \ldots\ldots + \alpha_m'\xi_m$, where

$$\alpha_1' \equiv \alpha\alpha_1 - \beta\alpha_2, \quad \alpha_2' \equiv \beta\alpha_1 + \alpha\alpha_2, \quad \alpha_j' \equiv \alpha_j. \qquad (j = 3, \ldots\ldots, m)$$

Of the $3^n \pm 1$ sets of values in the $GF[3^n]$ satisfying

$$\alpha^2 + \beta^2 = 1,$$

at most two give the same value to α_1' and hence at most four make $\overset{2}{\alpha_1'} = 1$. Hence, if $n > 1$, we can avoid the case $\alpha_1^2 = 1$. For $p^n = 3$, we may take

$$S = CW_{1234}W_{1567} \cdots\cdots W_{1\,3k-1\,3k\,3k+1},$$

where C is derived from the C_i and W is defined in §6. There remains for consideration the case in which, for example,*

$$\alpha_1^2 + \alpha_2^2 + \frac{1}{\mu}\alpha_m^2 \neq 0.$$

The treatment for a case like $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \neq 0$ is quite similar, taking $\mu = 1$.

We have proven that, for every set of solutions of

$$\alpha^2 + \beta^2 + \frac{1}{\mu}\gamma^2 = 1, \tag{6}$$

there exists a substitution Σ of the group

$$\xi_1' = \alpha\xi_1 + \beta\xi_2 + \gamma\xi_m, \quad \xi_2' = \alpha'\xi_1 + \beta'\xi_2 + \gamma'\xi_m, \quad \xi_m' = \alpha''\xi_1 + \beta''\xi_2 + \gamma''\xi_m,$$

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*For the case $p^n = 5$, $m \geqq 4$, $\mu =$ not-square, it would appear that the generator R were necessary in addition to the C_i and $O_{i,j}^{\alpha,\beta}$. We can, however, express R in terms of the generators

$$O_{i,m}^{2,1}: \begin{cases} \xi_i' = 2\xi_i + \xi_m \\ \xi' = 3\xi_i + 2\xi_m \end{cases},$$

leaving invariant $\xi_1^2 + \xi_2^2 + \ldots\ldots + \xi_{m-1}^2 + 3\xi_m^2$. Indeed,

$$R = O_{1m}O_{2m}O_{3m}^{-1}O_{1m}O_{2m}^{-1}O_{3m}^{-1}.$$

which therefore satisfies the relation (6) and the following:

$$\alpha'^2 + \beta'^2 + \frac{1}{\mu}\gamma'^2 = 1, \quad \alpha^2 + \alpha'^2 + \mu\alpha''^2 = 1, \quad \alpha\beta + \alpha'\beta' + \mu\alpha''\beta'' = 0, \text{ etc.}$$

If there be a substitution S' in our group which replaces ξ_1 by

$$\alpha_1'\xi_1 + \alpha_2'\xi_2 + \alpha_m'\xi_m + \sum_{j=3}^{m-1} \alpha_j\xi_j,$$

where

$$\begin{aligned} \alpha_1' &= \alpha\alpha_1 + \beta\alpha_2 + \frac{\gamma}{\mu}\alpha_m, \\ \alpha_2' &= \alpha'\alpha_1 + \beta'\alpha_2 + \frac{\gamma'}{\mu}\alpha_m, \\ \alpha_m' &= \mu\alpha''\alpha_1 + \mu\beta''\alpha_2 + \gamma''\alpha_m, \end{aligned}$$

then the group will contain $\Sigma S'$ which replaces ξ_1 by

$$\sum_{j=1}^{m} \alpha_j\xi_j.$$

The proposition is therefore true for the quantities α_j if true for α_1', α_2', α_m' $\alpha_4, \alpha_5, \ldots, \alpha_{m-1}$. We may thus make our proof by induction from $m-1$ to m by showing that it is possible to choose α, β, γ among the sets of solutions of (6) in such a way that $\alpha_1' = 0$. We may suppose that $\alpha_1 \neq 0$, since otherwise the proposition is already proven.

If $\alpha_1^2 + \alpha_2^2 = 0$, then $\alpha_2 \neq 0$. From $\frac{1}{\mu}\alpha_m^2 = 1$, it follows that μ is a square, say $\mu = 1$. Then the values

$$\alpha = \frac{-\alpha_m}{2\alpha_1}, \quad \beta = \frac{-\alpha_m}{2\alpha_2}, \quad \gamma = 1$$

satisfy (6) and make $\alpha_1' = 0$.

If $\alpha_1^2 + \alpha_2^2 \neq 0$, the condition (6) combines with $\alpha_1' = 0$ to give a single condition for β and γ:

$$\left(\beta\alpha_2 + \frac{\gamma}{\mu}\alpha_m\right)^2 + \alpha_1^2\left(\beta^2 + \frac{1}{\mu}\gamma^2\right) = \alpha_1^2.$$

Multiplying this by $\alpha_1^2 + \alpha_2^2$, it may be given the form

$$\left\{\beta\,(\alpha_1^2 + \alpha_2^2) + \frac{\alpha_2\alpha_m}{\mu}\gamma\right\}^2 + \frac{\gamma^2\alpha_1^2}{\mu}\left(\alpha_1^2 + \alpha_2^2 + \frac{\alpha_m^2}{\mu}\right) = \alpha_1^2\,(\alpha_1^2 + \alpha_2^2).$$

Since the coefficient of γ^2 is not zero, this equation has (by §3) $p^n \pm 1$ sets of solutions β, γ in the $GF[p^n]$.

13. *Note:* For the case $p^n = 3$, $m \overline{\gtrless} 4$, $\mu = -1$, it is readily seen that, instead of the additional generator V, we may take the more symmetrical substitution of period six:

$$X: \begin{cases} \xi_i' = \phantom{\xi_i +{}} \xi_j + \xi_k + \xi_m, \\ \xi_j' = \xi_i \phantom{{}+ \xi_j} + \xi_k + \xi_m, \\ \xi_k' = \xi_i + \xi_j + \phantom{\xi_k +{}} \xi_m, \\ \xi_m' = \xi_i + \xi_j + \xi_k - \xi_m, \end{cases}$$

where $(XC_1C_2C_3)^2 = 1$, $X^3 = C_1C_2C_3C_4$.

Structure of the Group $G^{(s)}_{m,p^n}$, §§14–32.

14. The substitutions of $G^{(s)}_{m,p^n}$ of determinant unity form a subgroup G of index 2. It is extended by C_1 to the total group.

By §3, there are $p^n - \varepsilon$ solutions α, β in the $GF[p^n]$ of

$$\alpha^2 + \frac{1}{\mu}\beta^2 = 1,$$

where $\varepsilon = +1$ or -1 according as $-\frac{1}{\mu}$ is a square or a not-square in the field. Hence the substitutions $O^{\alpha,\beta}_{i,j}$ which leave $\xi_i^2 + \mu\xi_j^2$ invariant and have the determinant unity form a group O_{ij} of order $p^n - \varepsilon$. Moreover, its substitutions are commutative; indeed

$$O^{\alpha',\beta'}_{i,j} O^{\alpha,\beta}_{i,j}: \begin{cases} \xi_i' = \left(\alpha\alpha' - \frac{\beta\beta'}{\mu}\right)\xi_i + (\alpha\beta' + \alpha'\beta)\xi_j, \\ \xi_j' = -\left(\frac{\alpha\beta' + \alpha'\beta}{\mu}\right)\xi_i + \left(\alpha\alpha' - \frac{\beta\beta'}{\mu}\right)\xi_j \end{cases}$$

is unaltered if we interchange α with α', β with β'. We shall use a notation for the square of such a substitution;

$$Q^{\alpha,\beta}_{i,j} \equiv (O^{\alpha,\beta}_{i,j})^2: \begin{cases} \xi_i' = (2\alpha^2 - 1)\xi_i + 2\alpha\beta\xi_j, \\ \xi_j' = -2\frac{\alpha\beta}{\mu}\xi_i + (2\alpha^2 - 1)\xi_j. \end{cases}$$

The substitutions $Q^{\alpha}_{i,j}$
Indeed, we can have

if and only if $\alpha' = \pm \alpha$
For our group G

if $i, j < m$ or if $i < j =$

if $j = m = s + 1$. The
Q_{ij} if i, $j < m$, while
If T_{ij} denote the
$i < j = m = s$, but not
$m = s$, if and only if 2

15. Let ρ, σ be
belong to the group Q

serves to extend the gr

Similarly, for $s =$
that $O^{\sigma}_{1,m}$ does not belo

serves to extend the gr
ν being a not-square.

16. *For* $p^n > 5$,
follows:

$H \equiv \{ Q$

where α, β *take all the*

contains half of the subs

Indeed, every sub

belong to H. Further, M_{ij} is commutative wi
$i, j)$. Also

$$M_{ij}\, Q_{i,\,k}^{\alpha,\,\beta} \equiv M_{ij}\,(O_{i,\,k}^{\lambda,\,\mu})^2\, Q_{ik}^{\lambda,\,-\mu}\cdot Q_{i,\,k}^{\alpha,\,\beta}$$
$$= (M_{ij}\, O_{i,\,k}^{\lambda,\,\mu})(Q_{i,\,k}^{\lambda,\,-\mu}\, Q_{i,\,k}^{\alpha,\,\beta})\, Q_{i,\,k}^{\lambda,\,\mu} = h' M_{ik}$$

gs to H), provided we take $\lambda,\ \mu = \rho,\ \sigma$ when i,
out take $\lambda,\ \mu = \varkappa,\ \tau$ when $i < k = m = s + 1$. Henc
e $h'' M_{r,\,s}$, where h'' belongs to H. If $s \gtreqqless 2$, we have th

$$M_{rs} = M_{rs}\, M_{21}\, M_{12} = h_1\, M_{12}.$$

stitution of G may be given one of the two forms, h
to H.

ases investigated (see §§30 and 49–55), it appears that
and hence of index two under it.

$= 5,\ m = s \gtreqqless 3$, the group

$$\equiv \{ C_i C_j,\quad T_{ij}\, T_{ik},\quad (i, j, k = 1, \ldots, m),\quad R\}$$

nder G. Indeed, 2 being a not-square modulo 5, T_{1}
We readily see that $T_{12} C_1$ is commutative with the
ansforms R into $C_2 C_3 R\, T_{12}\, T_{13}\, C_2 C_3$.
$m = s > 3$, the group

$$\equiv \{ C_i C_j,\quad T_{ij}\, T_{ik},\quad (i, j, k = 1, \ldots, m),\quad W\}$$

nder G. Here also $T_{12} C_1$ is not in the group Q_{12} an
; for example, it transforms W into $W^2 C_1 C_2$.
$m = s = 3$, the group of order twelve

$$[1,\ C_i C_j\ (\text{three}),\quad T_{ij}\, T_{ik}\ (\text{two}),\quad T_{ij}\, T_{ik} C_r C_s\ (\text{six})]$$

$_{12} C_1$ to the group G of order 24.
, $m = 3$, $s = 2$, the group leaving $\xi_1^2 + \xi_2^2 - \xi_3^2$ inv
at leaving $\xi_1^2 + \xi_2^2 + \xi_3^2$ by transforming by the substitu

$$O:\quad \xi_1' = \xi_1 - \xi_2,\quad \xi_2' = \xi_1 + \xi_2.$$

transforms $C_1 C_2,\ C_1 C_3,\ C_2 C_3,\ T_{12}\, T_{23},\ T_{13}\, T_{23}$ into re

C_1C_2, $T_{12}C_1C_2C_3$, $T_{12}C_3$, V and $V^2 \equiv V^{-1}$. Hence O transforms the group H of the last paragraph into

$$H \equiv [V^i, \quad V^iC_1C_2, \quad V^iT_{12}C_3, \quad V^iT_{12}C_1C_2C_3]. \qquad (i = 0, 1, 2)$$

For $p^n = 3$, $m > 3$, $s = m - 1$, the group generated as follows:

$$H \equiv \{ C_iC_j, \quad T_{ij}C_m, \quad (i, j = 1, \ldots, m-1), \quad V_{1,2,m} \}$$

is of index two under G and is extended to G by the substitution $T_{12}C_1$. The latter transforms $V_{1,2,m}$ into

$$V_{1,2,m}^2 C_1C_2.$$

18. Theorem: *When G is the orthogonal group (viz. $s = m$), the squares of its substitutions generate the group H.* Indeed, the squares of

$$O_{1,2}^{\alpha,\beta}, \quad O_{1,2}^{\alpha,\beta}T_{13}C_1C_2C_3, \quad O_{1,2}^{\alpha,\beta}T_{13}T_{24}$$

are respectively

$$O_{1,2}^{\alpha,\beta}, \quad O_{1,2}^{\alpha,\beta}O_{3,2}^{\alpha,\beta}, \quad O_{1,2}^{\alpha,\beta}O_{3,4}^{\alpha,\beta}.$$

For $p^n > 5$, H is generated by substitutions of these three types.

For $p^n = 5$ or 3, we have respectively

$$(RC_1C_2)^2 = T_{13}\,T_{23}C_1C_2RC_1C_3, \quad W^2 = W^{-1},$$

so that we obtain the necessary additional generators R or W respectively.

19. Every linear homogeneous substitution on m indices is commutative with

$$C \equiv C_1C_2 \ldots C_m: \quad \xi_i' = -\xi_i, \qquad (i = 1, \ldots, m)$$

of determinant $(-1)^m$. If m be odd, C does not belong to H. If m be even and $s = m$, C belongs to H. If m be even and $s = m - 1$, it seems probable that C does not belong to H, since it serves to extend H to G [see §§49–53 for the cases $m = 6$ and $m = 4$].

Suppose that H has an invariant subgroup I containing a substitution

$$S: \quad \xi_i' = \sum_{j=1}^{m} \alpha_{ij}\xi_j, \qquad (i = 1, \ldots, m)$$

nor C. We will prove that I coincides with H when m is ;, $m = 3$ being an exception, and when m is even and > 2, $= 4$ being an exception. The method of proof consists in l its transformed by substitutions in H a substitution belong- g at most three indices. Then I will contain all such substi- bgroup of H affecting only three indices is simple, aside from t then follows that I coincides with H.

'e substitution S be of the form

$$\xi_i' = \alpha_{ii}\xi_i, \qquad (i = 1 \ldots . m) \qquad (7)$$

$= 1$, it is merely a product of an even number of the C_i's, ıes as C_k are lacking, since S is neither the identity nor t if $S = C_i C_j C_r C_s \ldots .$, its transformed by $T_{ij} T_{ik}$, belonging $C_j C_r C_s \ldots .$ Hence I contains

$$S'S^{-1} = C_k C_i .$$

'orm (7), we may assume that $\alpha_{12}, \alpha_{13}, \ldots . , \alpha_{1m}$ are not all ' or its reciprocal will have at least one α_{ij} $(i < j)$ different ɔrming by $T_{1j} T_{1i}$, we have a substitution in I replacing ξ_1 by

$$\alpha_{ii}\xi_1 + \alpha_{ij}\xi_i + \ldots .$$

If $m > 4$, *the group I contains a substitution not the identity in*

μ a not-square ν when $s = m - 1$ and the square unity when we transform* S by

$$O_{234m} \begin{cases} \xi_2' = \alpha\xi_2 + \beta\xi_3 + \gamma\xi_4 + \delta\xi_m, \\ \xi_3' = \alpha'\xi_2 + \beta'\xi_3 + \gamma'\xi_4 + \delta'\xi_4, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

$$\left(\alpha^2 + \beta^2 + \gamma^2 + \frac{1}{\mu}\,\delta^2 = 1, \text{ etc.}\right)$$

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long to H, its product by O_{4m} will belong to H, O_{4m} being suitably chosen. A ɜtood to apply in the succeeding paragraphs.

The coefficients in the resulting substitution are

$$\alpha'_{11}=\alpha_{11}, \quad \alpha'_{12}=\alpha\alpha_{12}+\beta\alpha_{13}+\gamma\alpha_{14}+\frac{\delta}{\mu}\alpha_{1m}, \text{ etc.}$$

As in §12, we can determine $\alpha, \beta, \gamma, \delta$ so that $\alpha'_{12}=0$, unless perhaps in the case for which

$$p=3, \quad \mu=1, \quad \alpha_{12}^2=\alpha_{13}^2=\alpha_{14}^2=\alpha_{1m}^2.$$

In this case the transformed of S by KW_{834m}, K being a suitable product formed from C_2, C_3, C_4, C_m, will give a substitution belonging to I in which $\alpha'_{12}=\alpha'_{14}=\alpha'_{1m}=0$.

22. Theorem: *If $m>4$, the group I contains a substitution affecting only two indices or else a substitution in which α_{12} has an arbitrary value τ in the $GF[p^n]$.*

In virtue of §20, it remains to consider the case in which not every α_{1j} $(j=2, \ldots., m)$ is zero.

If $\alpha_{1m}\neq 0$, $\alpha_{1j}=0$ $(j=2, \ldots., m-1)$, we transform S by $O_{2,\,3,\,m}^{\alpha,\,\beta,\,\gamma}$, obtaining a substitution S' in which

$$\alpha'_{12}=\alpha\alpha_{12}+\beta\alpha_{13}+\frac{\gamma}{\mu}\alpha_{1m}.$$

Taking $\gamma=\dfrac{\mu\tau}{\alpha_{1m}}$ and α, β such that $\alpha^2+\beta^2+\dfrac{\gamma^2}{\mu}=1$, we have in S' a substitution belonging to I and having $\alpha'_{12}=\tau$.

If $\alpha_{12}, \alpha_{13}, \ldots., \alpha_{1m-1}$ are not all zero, we may make $\alpha_{12}=0$ by §21, and suppose that, for example, $\alpha_{14}\neq 0$. Transforming S by $O_{2,\,3,\,4}^{\alpha,\,\beta,\,\gamma}$, we obtain a substitution S' in which

$$\alpha'_{11}\equiv\alpha_{11}, \quad \alpha'_{12}\equiv\alpha\alpha_{12}+\beta\alpha_{13}+\gamma\alpha_{14}.$$

To prove that there exists in the $GF[p^n]$ a set of solutions of

$$\beta\alpha_{13}+\gamma\alpha_{14}=\tau, \quad \alpha^2+\beta^2+\gamma^2=1,$$

we combine them into the single relation

$$\beta^2(\alpha_{13}^2+\alpha_{14}^2)-2\beta\tau\alpha_{13}+\alpha^2\alpha_{14}^2=\alpha_{14}^2-\tau^2.$$

For $\alpha_{13}^2+\alpha_{14}^2=0$, and therefore $\alpha_1\neq 0$, a set of solutions is given by $\alpha=0$ when $\tau\neq 0$ and by $\alpha=1$, $\beta=0$ when $\tau=0$.

For $\alpha_{13}^2 + \alpha_{14}^2 \neq 0$, there exist solutions of the equivalent equation of condition

$$\{\beta(\alpha_{13}^2 + \alpha_{14}^2) - \tau\alpha_{13}\}^2 + \alpha^2\alpha_{14}^2(\alpha_{13}^2 + \alpha_{14}^2) = \alpha_{14}^2(\alpha_{13}^2 + \alpha_{14}^2 - \tau^2).$$

23. Theorem: *From a substitution S of I in which α_{12} has an arbitrary value we can obtain one in which $1 - \alpha_{11}^2$ is a square, not zero, in the $GF[p^n]$.*

The required substitution belonging to I is the following:

$$S^{-1}C_1C_2SC_1C_2 \equiv S_\alpha C_1C_2,$$

where S_α denotes the substitution of period two,

$$\xi_i' = \xi_i - 2\alpha_{i1}\left(\sum_{j=1}^{m-1}\alpha_{j1}\xi_j + \mu\alpha_{m1}\xi_m\right) - 2\alpha_{i2}\left(\sum_{j=1}^{m-1}\alpha_{j2}\xi_j + \mu\alpha_{m2}\xi_m\right).$$
$$(i = 1, 2, \ldots, m)$$

The coefficient of ξ_1 in ξ_1' in the product $S_\alpha C_1C_2$ is

$$\bar{\alpha}_{11} \equiv -(1 - 2\alpha_{11}^2 - 2\alpha_{12}^2).$$

Since α_{12} is arbitrary, $\bar{\alpha}_{11}$ takes $(p^n + 1)/2$ distinct values in the field. But, by §4, the number of squares ξ^2 for which $1 - \xi^2$ is a not-square is $(p^n - 1)/4$ or $(p^n - 3)/4$ according as -1 is a square or not-square in the $GF[p^n]$, a result which follows immediately since $\nu\eta^2 + \xi^2 = 1$ has $p^n \pm 1 - 2$ sets of solutions for which the not-square $\nu\eta^2 \neq 0$. Hence $1 - \bar{\alpha}_{11}^2$ takes at least one value other than a not-square. The theorem is therefore proven unless $\bar{\alpha}_{11}^2 = 1$. But if we start from a substitution in which $\alpha_{11}^2 = 1$, we derive a substitution in which $\bar{\alpha}_{11} = 1 + 2\alpha_{12}^2$, and therefore

$$1 - \bar{\alpha}_{11}^2 = -4(\alpha_{12}^2 + 1)\alpha_{12}^2,$$

which, by choice of α_{12}, can be made a square when $p^n \neq 5$. Indeed, we can determine $\alpha_{12} \neq 0$ and σ such that $-1 - \alpha_{12}^2 = \sigma^2 \neq 0$; for there are $p^n - \varepsilon$ sets of solutions in the $GF[p^n]$ of

$$-1 = \alpha_{12}^2 + \sigma^2,$$

ε being ± 1 according as -1 is a square or a not-square. Hence there are $p^n - 5$ or $p^n + 1$ sets of solutions in which $\alpha_{12} \neq 0$, $\sigma \neq 0$.

$$\bar{\alpha}_{11} = -(1 + 2 - 2\alpha_{12}^2) = 0 \text{ for } \alpha_{12} = 2.$$

24. Theorem: If $m = 4$, $s = 3$, *the group* I *contains a substitution in which* α_{12} *is an arbitrary mark in the* $GF[p^n]$, *or else a substitution affecting only two indices.*

We have the relation between the coefficients of S,

$$\alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 + \frac{1}{\nu}\alpha_{14}^2 = 1. \qquad (\nu = \text{not-square})$$

(1). Suppose first that $\alpha_{11}^2 = 1$. Transforming S by

$$O_{2,\,4}^{\alpha,\,-\beta}, \qquad \left(\alpha^2 + \frac{1}{\nu}\beta^2 = 1\right)$$

we obtain a substitution replacing ξ_1 by

$$\alpha_{11}\xi_1 + \left(\alpha\alpha_{12} - \frac{\beta}{\nu}\alpha_{14}\right)\xi_2 + \alpha_{13}\xi_3 + (\beta\alpha_{12} + \alpha\alpha_{14})\,\xi_4.$$

If $\alpha_{12}^2 + \frac{1}{\nu}\alpha_{14}^2$ is a not-square and therefore $\alpha_{13} \neq 0$, we can make $\alpha_{12}' = 0$ by taking

$$\alpha = \frac{\beta}{\nu}\,\frac{\alpha_{14}}{\alpha_{12}}, \quad \beta\left(\alpha_{12}^2 + \frac{1}{\nu}\alpha_{14}^2\right) = \nu\alpha_{12}^2.$$

From a substitution in which $\alpha_{13}^2 + \frac{1}{\nu}\alpha_{14}^2 = 0$, $\alpha_{12} = 0$, we can obtain, by transformation by $O_{3,\,4}^{\alpha,\,\beta}$, a substitution in which α_{13}' has an arbitrary value τ. Indeed, the values

$$\alpha = \frac{\tau^2 + \alpha_{13}^2}{2\tau\alpha_{13}}, \quad \beta = \frac{\nu(\tau^2 - \alpha_{13}^2)}{2\tau\,\alpha_{14}}$$

make

$$\alpha_{13}' \equiv \alpha\alpha_{13} + \frac{\beta}{\nu}\alpha_{14} = \tau, \quad \alpha^2 + \frac{1}{\nu}\beta^2 = 1.$$

If $\alpha_{12}^2 + \frac{1}{\nu}\alpha_{14}^2$ is a square, we can make $\alpha_{14}' = 0$ by taking

$$\alpha^2\left(\alpha_{12}^2 + \frac{1}{\nu}\alpha_{14}^2\right) = \alpha_{12}^2, \quad \beta = \frac{-\alpha\alpha_{14}}{\alpha_{12}}.$$

With $\alpha_{14} = 0$, we have $\alpha_{12}^2 + \alpha_{13}^2 = 0$. Transforming by $O_{2,3}^{\alpha,\beta}$ we can, as above, make $\alpha_{12} = \tau$, an arbitrary mark $\neq 0$.

The substitution $S^{-1}C_1C_2SC_1C_2$, as shown in §23, has the coefficient $\bar{\alpha}_{11} \equiv 1 + 2\alpha_{12}^2$, since $\alpha_{11}^2 = 1$. Hence $\bar{\alpha}_{11}$ will reduce to ± 1 only when $\alpha_{12}^2 = 0$ or -1. Since we can choose $\tau \neq 0$ such that $\tau^2 \neq -1$, we have a substitution belonging to I in which $\bar{\alpha}_{11}^2 \neq 1$, a case next treated.

(2). Suppose, however, that $\alpha_{11}^2 \neq 1$. Then, since

$$\alpha_{12}^2 + \alpha_{13}^2 + \frac{1}{\nu}\alpha_{14}^2 \neq 0,$$

we can determine a substitution O_{234}, as in §12, which will transform S into a substitution having $\alpha_{12} = 0$. If $\alpha_{13} = 0$, we can at once make $\alpha_{12}' = \tau$, as in §22. If $\alpha_{13} \neq 0$, we transform S by $O_{2,3,4}^{\alpha,\beta,\gamma}$ and make

$$\alpha_{12}' \equiv \alpha\alpha_{12} + \beta\alpha_{13} + \frac{\gamma}{\nu}\alpha_{14} = \tau, \quad \alpha^2 + \beta^2 + \frac{\gamma^2}{\nu} = 1.$$

These relations combine, on eliminating β, into

$$\left\{\gamma\left(\alpha_{13}^2 + \frac{1}{\nu}\alpha_{14}^2\right) - \tau\alpha_{14}\right\}^2 + \nu\alpha_{13}\left(\alpha_{13}^2 + \frac{1}{\nu}\alpha_{14}^2\right)\alpha^2 = \nu\,\alpha_{13}^2\left(\alpha_{13}^2 + \frac{1}{\nu}\alpha_{14}^2 - \tau^2\right),$$

which has $p^n \pm 1$ sets of solutions γ, α in the $GF[p^n]$; indeed, $\alpha_{13}^2 + \frac{1}{\nu}\alpha_{14}^2 \neq 0$.

25. Theorem: *If $m > 4$ or if $m = 4$, $s = 3$, the group I contains a substitution not the identity and replacing ξ_1 by $\alpha_{11}\xi_1 + \alpha_{12}\xi_2$.*

By a repeated application of §21, we can suppose that

$$\alpha_{1m-1} = \alpha_{1m-2} = \ldots. = \alpha_{15} = \alpha_{14} = 0.$$

By §§22-24, we can suppose that I contains a substitution affecting only ξ_1 and ξ_2, when the theorem is proven, or a substitution in which $1 - \alpha_{11}^2 =$ square. In the latter case,

$$\alpha_{12}^2 + \alpha_{13}^2 + \frac{1}{\mu_J}\alpha_{1m}^2 = 1 - \alpha_{11}^2 \neq 0,$$

so that by §12 we can make $\alpha_{13} = 0$. We then have

$$\alpha_{13}^2 + \frac{1}{\mu}\alpha_{1m}^2 = 1 - \alpha_{11}^2 = \text{square}.$$

Then, as in §24 we can make $\alpha_{1m} = 0$, when the theorem is proven.

The substitution reached is neither the identity nor $C_1C_2 \dots . C_m$. Indeed, $1 - \alpha_{11}^2 \neq 0$. For the case in which S was of the form treated in §20, the substitution reached was C_iC_k.

26. Theorem: *If $m > 4$ or if $m = 4$, $s = 3$, the group I contains a substitution leaving ξ_1 fixed and not the identity.*

The substitution obtained in §25 is evidently a product $O_{1,2}^{\alpha_{11},\alpha_{12}} S_1$, where S_1 leaves ξ_1 fixed.

If S be not commutative with C_1C_2, I contains

$$S^{-1}\,C_1C_2SC_1C_2 = S_1^{-1}\,C_1C_2S_1C_1C_2 = S_1^{-1}\,C_2S_1C_2 \neq 1,$$

which evidently leaves ξ_1 fixed.

If S be commutative with C_1C_2, S_1 is commutative with C_2 and therefore replaces ξ_2 by $\pm\xi_2$. If S_1 be commutative with every $Q_{i,j}^{\alpha,\beta}$ $(i, j = 3, \dots . , m)$, it has, by §28, the form

$$\xi_1' = \xi_1, \quad \xi_2' = \pm\,\xi_2, \quad \xi_i' = \lambda\xi_i, \qquad (i = 3, \dots . , m)$$

where $\lambda^2 = 1$. If then $O_{1,2}^{\alpha_{11},\alpha_{12}}$ be either the identity or C_1C_2, S is of the form treated in §20. If $O_{1,2}$ be not of either form, its square is not the identity, so that S^2 is a substitution of I not the identity and leaving $\xi_3, \dots . , \xi_m$ fixed. If, however, S_1 be not commutativve with $Q_{3,4}^{\alpha,\beta}$, for example, I will contain

$$S^{-1}Q_{3,4}^{-1}SQ_{3,4} \equiv S_1^{-1}Q_{3,4}^{-1}S_1Q_{3,4} \neq 1,$$

which evidently leaves fixed ξ_1 and ξ_2.

27. Theorem: *If $m > 4$ or if $m = 4$, $s = 3$, the group I contains a substitution, not the identity, affecting at most three indices.*

If $s = m - 1$, a repeated application of the previous theorem gives a substitution, not the identity, belonging to I, and affecting only three indices.

If $s = m > 4$, we obtain by the same theorem a substitution

$$\xi_i' = \sum_{j=1}^{4} \gamma_{ij}\xi_j, \qquad (i = 1, 2, 3, 4)$$

not the identity and belonging to I. By §20, we may suppose that $\gamma_{12} \neq 0$. We can make $\gamma_{11}^2 \neq 1$. For, if $\gamma_{11}^2 = 1$, we transform S by $O_{2,\,3}^{\alpha,\,\beta}$, giving a substitution S' in which

$$\gamma'_{11} = \gamma_{11}, \quad \gamma'_{12} = \alpha\gamma_{12} + \beta\gamma_{13}, \quad \gamma'_{13} = -\beta\gamma_{12} + \alpha\gamma_{13}, \quad \gamma'_{14} = \gamma_{14}.$$

At most two of the $p^n \pm 1$ sets of solutions of $\alpha^2 + \beta^2 = 1$ give the same value to γ'_{13}. Hence, if $p^n > 5$, there are at least $4 = \frac{1}{2}(9 - 1) = \frac{1}{2}(7 + 1)$ values of γ'_{13}, and therefore values for which γ'^{2}_{13} is neither zero nor -1. Then in the substitution

$$\bar{S} \equiv S'^{-1} C_1 C_2 S' C_1 C_2,$$

the coefficient

$$\bar{\gamma}_{11} \equiv -(1 - 2\gamma'^{2}_{11} - 2\gamma'^{2}_{12})$$

has a value different from ± 1.

For $p^n = 3$, we have by hypothesis $\gamma_{11}^2 = 1$, $\gamma_{13}^2 = 1$. The substitution $S^{-1} C_1 C_2 S C_1 C_2$ will therefore have $\bar{\gamma}_{11} = 0$.

For $p^n = 5$, the equation $\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{14}^2 = 0$ requires that one of the three squares be zero, another $+1$ and the third -1, since all are not zero. Transforming by a substitution of the form $T_{23} T_{24}$ or $T_{23} T_{34}$, if a transformation be necessary at all, we may take $\gamma_{12}^2 = 1$, $\gamma_{13}^2 = -1$, $\gamma_{14}^2 = 0$. Then $\bar{\gamma}_{11} = 3$.

In every case we have in I a quaternary substitution S' in which $\gamma_{11}^2 \neq 1$. It is therefore not commutative with C_1. Hence, m being > 4, I contains

$$S'^{-1} C_1 C_5 S' C_1 C_5 \equiv S'^{-1} C_1 S C_1 \equiv S_\gamma C_1 \neq 1,$$

where S_γ denotes the substitution

$$\xi'_i = \xi_i - 2\gamma_{i1} \sum_{j=1}^{4} \gamma_{j1} \xi_j. \qquad (i = 1, \ldots, 4)$$

We may, by §12, make $\gamma_{41} = 0$, since we have

$$\gamma_{21}^2 + \gamma_{31}^2 + \gamma_{41}^2 = 1 - \gamma_{11}^2 \neq 0.$$

We therefore have a substitution in I affecting only three indices and different from the identity.

28. Lemma: *If a substitution S of G be commutative with $O_{1,\,m}^{\alpha,\,\beta} \neq 1$, it breaks up into the product of a substitution affecting ξ_1 and ξ_m only and a substitution affecting $\xi_2, \ldots, \xi_{m-1}$ only.*

Indeed, the conditions for the identity $O_{1,\,m}^{\alpha,\,\beta} S \equiv S O_{1,\,m}^{\alpha,\,\beta}$ are:

$$\text{(a)}\quad \beta\alpha_{11} = \beta\alpha_{mm}, \quad \beta\alpha_{m1} = -\frac{\beta}{\mu}\alpha_{1m};$$

$$\left.\begin{array}{ll}\text{(b)}\quad (\alpha-1)\alpha_{1j} + \beta\alpha_{mj} = 0, & -\dfrac{\beta}{\mu}\alpha_{1j} + (\alpha-1)\alpha_{mj} = 0, \\ \text{(c)}\quad (\alpha-1)\alpha_{j1} - \dfrac{\beta}{\mu}\alpha_{jm} = 0, & \beta\alpha_{j1} + (\alpha-1)\alpha_{jm} = 0,\end{array}\right\} \quad (j = 2, \ldots\ldots, m-1)$$

Since $\alpha^2 + \frac{1}{\mu}\beta^2 = 1$ and $\alpha \neq 1$, $O_{1,\,2}^{\alpha,\,\beta}$ not being the identity, we have for the determinant of the pair of equations (b) and likewise for the pair (c),

$$(\alpha-1)^2 + \frac{1}{\mu}\beta^2 = 2 - 2\alpha \neq 0.$$

Hence must

$$\alpha_{1j} = \alpha_{mj} = \alpha_{j1} = \alpha_{jm} = 0. \qquad (j = 2, \ldots\ldots, m-1)$$

Hence $S \equiv S_{1m} S_{23 \ldots\ldots m-1}$, where S_{1m} affects only ξ_1 and ξ_m, and $S_{23 \ldots\ldots m-1}$ affects only $\xi_2, \ldots\ldots, \xi_{m-1}$. Since S leaves invariant

$$\xi_1^2 + \xi_2^2 + \ldots\ldots + \xi_{m-1}^2 + \mu\xi_m^2,$$

S_{1m} must leave $\xi_1^2 + \mu\xi_m^2$ invariant, and hence be either $O_{1,\,m}^{\alpha_{11},\,\alpha_{1m}}$ or its product by C_1. The latter case is evidently excluded except when $O_{1,\,m}^{\alpha,\,\beta} \equiv C_1 C_m$. Indeed, with this exception, $\beta \neq 0$ so that (a) gives new conditions.

A like result follows if S be commutative with $O_{i,\,j}^{\alpha,\,\beta}$ where $i, j < m$.

29. Theorem: *If* $m > 4$ *or if* $m = 4$, $s = 3$, *the group* I *coincides with* H.

For $p^n > 3$, the subgroup of H which affects three indices only is by §§30–31 a simple group. Since I contains one of the substitutions of this simple group, it contains all. Transforming them by the substitutions $T_{ij}T_{ik}$, belonging to H, we obtain every substitution of H affecting three indices. Hence, for $p^n > 3$, I contains all the generators of H.

For $p^n = 3$, $m = s > 4$, I contains one of the substitutions affecting three indices ξ_1, ξ_2, ξ_3, and not the identity, which by §17 are the following eleven:

$$C_i C_j, \quad T_{ij} T_{ik}, \quad T_{ij} T_{ik} C_r C_s. \qquad (i, j, k, r, s = 1, 2, 3)$$

If it contain one of the last two types, I contains its transformed by C_iC_j, viz.

$$T_{ij}T_{ik}C_iC_k \text{ or } T_{ij}T_{ik}C_rC_s \,.\, C_iC_k.$$

Hence, in every case, I contains C_iC_k, and therefore also every product of two C_i's. Hence I contains

$$T_{12}T_{34} = W^{-1}C_3C_4W, \quad W = W^{-1}C_1C_5WC_1C_5.$$

Since the alternating group on $m > 4$ indices is simple, I contains every product $T_{ij}T_{kl}$. Hence $I \equiv H$.

For $p^n = 3$, $m > 3$, $s = m - 1$, the group I contains one of the substitutions, not the identity, of the group G_{12} leaving invariant $\xi_1^2 + \xi_2^2 - \xi_m^2$, which by §17 is the transformed by O of the group G'_{12}, leaving invariant $\xi_1^2 + \xi_2^2 + \xi_m^2$. We have just proven that any substitution of G'_{12} can be combined with its transformed (by substitutions of G'_{12}) so as to give C_1C_2. The same result holds for G_{12} since O transforms C_1C_2 into itself. Hence I contains every C_iC_j $(i, j < m)$. But $V_{1,2,m}^{-1}$ transforms C_1C_2 into $T_{12}C_m$. Hence I contains every $T_{ij}C_m$ $(i, j < m)$. Finally, I contains $V_{1,2,m}$, since

$$V_{1,2,m}^{-1}(T_{12}C_2C_3C_m)^{-1}V_{1,2,m}(T_{12}C_2C_3C_m) = V_{1,2,m}C_1C_2.$$

Hence in this case also I coincides with H.

30. Theorem: *The ternary orthogonal group H in the $GF[p^n]$, $p > 2$, having the order $\frac{1}{2}p^n(p^{2n}-1)$, is simply isomorphic to the group Γ in the $GF[p^n]$ of linear fractional substitutions of determinant unity on one index.*

Let i be a root of the equation $\xi^2 = -1$, so that i belongs to the $GF[p^n]$ or to the $GF[p^{2n}]$ according as -1 is a square or not-square in the $GF[p^n]$.

Introduce in place of ξ_1, ξ_2, ξ_3 the new indices

$$\eta_1 \equiv -i\xi_1, \quad \eta_2 \equiv \xi_2 - i\xi_3, \quad \eta_3 \equiv \xi_2 + i\xi_3,$$

whence

$$\eta_2\eta_3 - \eta_1^2 \equiv \xi_1^2 + \xi_2^2 + \xi_3^2.$$

The orthogonal substitution

$$S: \quad \xi'_i = \sum_{j=1}^{3} \alpha_{ij}\xi_j \qquad (i = 1, 2, 3)$$

takes the form

$$S_1:\quad \begin{cases} \eta_1' = \alpha_{11}\eta_1 + \frac{1}{2}(\alpha_{13} - i\alpha_{12})\,\eta_2 - \frac{1}{2}(\alpha_{13} + i\alpha_{12})\,\eta_3, \\ \eta_2' = (\alpha_{31} + i\alpha_{21})\,\eta_1 + \frac{1}{2}(\alpha_{22} - i\alpha_{32} + i\alpha_{23} + \alpha_{33})\,\eta_2 + \frac{1}{2}(\alpha_{22} - i\alpha_{32} - i\alpha_{23} - \alpha_{33})\,\eta_3, \\ \eta_3' = (-\alpha_{31} + i\alpha_{21})\,\eta_1 + \frac{1}{2}(\alpha_{22} + i\alpha_{32} + i\alpha_{23} - \alpha_{33})\,\eta_2 + \frac{1}{2}(\alpha_{22} + i\alpha_{32} - i\alpha_{23} + \alpha_{33})\,\eta_3. \end{cases}$$

We proceed to prove that S_1 can be given the form

$$\begin{pmatrix} \alpha\delta + \beta\gamma & \alpha\gamma & \beta\delta \\ 2\alpha\beta & \alpha^2 & \beta^2 \\ 2\gamma\delta & \gamma^2 & \delta^2 \end{pmatrix}, \qquad [\alpha\delta - \beta\gamma = 1] \qquad (8)$$

where $\alpha, \beta, \gamma, \delta$ are complexes of the form $\rho + \sigma i$, ρ and σ being marks of the $GF[p^n]$. The proof will follow for the general substitution S of H, if proven for the generators of H. Indeed, denoting the substitution (8) by $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, we verify the composition formula,

$$\begin{bmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha\alpha' + \beta\gamma' & \alpha\beta' + \beta\delta' \\ \gamma\alpha' + \delta\gamma' & \gamma\beta' + \delta\delta' \end{bmatrix}.$$

Hence the product of two substitutions of the form (8) is again of the form (8), the composition being identical with that for linear fractional substitutions. Expressing the orthogonal substitution $O_{2,3}^{\alpha,\beta}$ in terms of the indices η_1, η_2, η_3, we obtain the substitution

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha + \beta i & 0 \\ 0 & 0 & \alpha - \beta i \end{pmatrix}, \qquad [(\alpha + \beta i)(\alpha - \beta i) = 1]$$

which need not be of the form (8); whereas its square $Q_{2,3}^{\alpha,\beta}$ is always of the form (8). The product $O_{1,3}^{\alpha,\beta}O_{2,3}^{\alpha,\beta}$ expressed in the indices η_i is

$$\begin{pmatrix} \alpha & -\frac{i\beta}{2} & -\frac{i\beta}{2} \\ -\beta i(\alpha + \beta i) & \frac{\alpha + 1}{2}(\alpha + \beta i) & \frac{\alpha - 1}{2}(\alpha + \beta i) \\ -\beta i(\alpha - \beta i) & \frac{\alpha - 1}{2}(\alpha - \beta i) & \frac{\alpha + 1}{2}(\alpha - \beta i) \end{pmatrix},$$

which is of the form (8), viz. in the above notation

$$\begin{bmatrix} \frac{1}{2}(\alpha + 1 + \beta i) & -\frac{1}{2}(\alpha - 1 + \beta i) \\ \frac{1}{2}(\alpha - 1 - \beta i) & \frac{1}{2}(\alpha + 1 - \beta i) \end{bmatrix}.$$

In particular we have $T_{12}T_{23}$ so expressed. For $T_{12}T_{13}$ we have

$$\begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ i & -i/2 & -i/2 \\ i & i/2 & i/2 \end{pmatrix} \equiv \begin{bmatrix} \frac{1-i}{2} & -\frac{(1-i)}{2} \\ \frac{1+i}{2} & \frac{1+i}{2} \end{bmatrix}.$$

For $p^n = 5$, we have for the generator R:

$$\begin{pmatrix} 1 & \frac{1}{2}(2-i) & -\frac{1}{2}(2+i) \\ 2+i & \frac{1}{2}.3 & \frac{1}{2}(1-2i) \\ -2+i & \frac{1}{2}(1+2i) & \frac{1}{2}3 \end{pmatrix} = \begin{bmatrix} -3 & 3-i \\ 3+i & 3 \end{bmatrix}.$$

Since H can be generated from the above substitutions, it follows that every substitution of H can be put into the form (8).

If -1 be a square, the coefficients α, β, γ, δ belong to the $GF[p^n]$, so that H is simply isomorphic to Γ.

If -1 be a not-square, α and δ, β and γ are conjugate imaginaries in i, so that H is simply isomorphic to the imaginary form* of the group Γ. But Γ is known† to be a simple group if $p^n > 3$.

Corollary. For $m = 3$, the group H does not coincide with G.

31. Theorem: *The subgroup H_ν of the group G_ν of all linear substitutions leaving $\xi_1^2 + \xi_2^2 + \nu\xi_3^2$ invariant is simple if $p^n > 3$.*

Since the substitution

$$O: \begin{cases} \xi_1' = \alpha\xi_1 - \beta\xi_2 \\ \xi_2' = \beta\xi_1 + \alpha\xi_2 \end{cases} \qquad (\alpha^2 + \beta^2 = \nu)$$

transforms $\xi_1^2 + \xi_2^2 + \nu\xi_3^2$ into $\nu(\xi_1^2 + \xi_2^2 + \xi_3^2)$, it transforms G_ν into the ternary orthogonal group G. Further, O transforms C_1C_3, which extends H_ν to G_ν, into $O_{1,2}^{\rho,\sigma}C_1C_3$, where

$$\rho = \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2}, \quad \sigma = \frac{2\alpha\beta}{\alpha^2 + \beta^2}, \quad \rho^2 + \sigma^2 = 1.$$

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*Moore, *A doubly-infinite system of simple groups*, Congress Mathematical Papers, 1893.

†Besides the proof by Moore, the theorem has been established by Burnside in the Proceedings of the London Mathematical Society, 1894, and by Dickson in the Annals of Mathematics, 1897.

The latter substitution serves to extend H to G; indeed $O^{\rho,\,g}_{1,\,2}$ is not in the group $Q_{1,\,2}$ since

$$\frac{1+\rho}{2} \equiv \frac{\alpha^2}{\alpha^2+\beta^2} = \frac{\alpha^2}{\nu}$$

is a not-square, and therefere ρ not of the form $2\,S^2-1$.

It follows that H_ν is simply isomorphic to H.

Linear homogeneous group in the Galois field of order 2^n defined by a quadratic invariant, §§32–48.

32. We will assume that the invariant

$$f \equiv \sum_{i \leqq j}^{i,\,j=1\,\ldots\,m} \alpha_{ij}\xi_i\xi_j$$

cannot be expressed as a quadratic function of fewer than m variables belonging to the $GF[2^n]$. It will be convenient to set $\alpha_{ji} \equiv \alpha_{ij}$.

Theorem: *We can determine a linear homogeneous substitution belonging to the $GF[2^n]$ which will transform f into one of the following forms*:

$$(m\ odd) \qquad \xi_1\xi_2+\xi_3\xi_4+\ldots\ldots+\xi_{m-2}\xi_{m-1}+\xi_m^2,$$

$$(m\ even) \quad \xi_1\xi_2+\xi_3\xi_4+\ldots\ldots+\xi_{m-3}\xi_{m-2}+\alpha\xi_{m-1}^2+\beta\xi_{m-1}\xi_m+\gamma\xi^2_m.$$

We first prove that, if $m \geqq 3$, f can be transformed into a quadratic form having $\alpha_{11}=0$. If every $\alpha_{ij}\ (i,j=1,\ldots\ldots,m;\ i \neq j)$ were zero, f would have the form

$$f \equiv (\sqrt{\alpha_{11}}\xi_1+\sqrt{\alpha_{22}}\xi_2+\ldots\ldots+\sqrt{\alpha_{mm}}\xi_m)^2.$$

This being contrary to our hypothesis, we may assume that $\alpha_{23} \neq 0$, for example. We may also suppose that $\alpha_{22} \neq 0$, since otherwise the transformed of f by $(\xi_1\xi_2)$ would have $\alpha_{11}=0$. The terms of f which involve ξ_2 may be written thus,

$$\alpha_{22}\xi_2^2+\xi_2(\alpha_{21}\xi_1+\alpha_{23}\xi_3+\alpha_{24}\xi_4+\ldots\ldots+\alpha_{2m}\xi_m).$$

Hence the inverse of the following substitution,

$$\begin{aligned} &\xi_3'=\alpha_{21}\xi_1+\alpha_{23}\xi_3+\alpha_{24}\xi_4+\ldots\ldots+\alpha_{2m}\xi_m,\\ &\xi_i'=\xi_i \qquad\qquad\qquad\qquad (i=1,\ldots\ldots,m;\ i \neq 3),\end{aligned}$$

will transform f into

$$\alpha_{22}\xi_2^2 + \xi_2\xi_3 + \Sigma\beta_{ij}\xi_i\xi_j,$$

summed for $i, j = 1, 3, 4, \ldots\ldots, m$; $i < j$. Applying the substitution

$$\xi_2' = \xi_2 + \lambda\xi_1, \quad \xi_i' = \xi_i, \qquad (i = 1, 3, 4, \ldots\ldots, m)$$

we obtain as the new coefficient of ξ_1^2 the function $\alpha_{22}\lambda^2 + \beta_{11}$, which may be made to vanish by determining λ.

We may therefore suppose that $\alpha_{11} = 0$ in our original function f. Since the α_{1j} are not all zero, we may assume that $\alpha_{12} \neq 0$. Applying to f the inverse of the substitution

$$\xi_2' = \alpha_{12}\xi_2 + \alpha_{13}\xi_3 + \ldots\ldots + \alpha_{1m}\xi_m, \quad \xi_i' = \xi_i \qquad (i = 1, 3, 4, \ldots\ldots, m)$$

we obtain the function

$$\xi_1\xi_2 + \sum_{i<j}^{i,j=2,\ldots\ldots,m} \gamma_{ij}\xi_i\xi_j.$$

Replacing $\xi_1 + \gamma_{22}\xi_2 + \gamma_{23}\xi_3 + \ldots\ldots + \gamma_{2m}\xi_m$ by ξ_1, we get

$$f' \equiv \xi_1\xi_2 + \sum_{i,j}^{3,\ldots\ldots,m} \delta_{ij}\xi_i\xi_j.$$

Similarly, if $m \geqq 5$, we can transform f' into

$$\xi_1\xi_2 + \xi_3\xi_4 + \sum_{i,j}^{5,\ldots\ldots,m} \varepsilon_{ij}\xi_i\xi_j.$$

The theorem follows by a simple induction.

33. Theorem: *For m even, the quadratic invariant can be reduced by a linear substitution in the $GF[2^n]$ to the form*

$$F_\lambda \equiv \xi_1\xi_2 + \xi_3\xi_4 + \ldots\ldots + \xi_{m-1}\xi_m + \lambda\xi_{m-1}^2 + \lambda\xi_m^2,$$

where $\lambda = 0$ or has any one of the values for which the form $\xi_{m-1}\xi_m + \lambda\xi_{m-1}^2 + \lambda\xi_{2m}^2$ is irreducible in the $GF[2^n]$.

If $\alpha\xi_{m-1}^2 + \beta\xi_{m-1}\xi_m + \gamma\xi_m^2$ be reducible, the form reached in §32 can evidently be reduced to F_0. In the contrary case, it can readily be given the form

$$\xi_1\xi_2 + \xi_3\xi_4 + \ldots\ldots + \xi_{m-3}\xi_{m-2} + \xi_{m-1}^2 + \xi_{m-1}\xi_m + \delta\xi_m^2,$$

δ being such a mark that the equation

$$\xi^2 + \xi + \delta = 0 \tag{9}$$

is irreducible in the $GF[2^n]$. It follows from (9) that

$$\xi^{2^n} = \xi + \delta + \delta^2 + \delta^4 + \ldots\ldots + \delta^{2^{n-1}}.$$

Hence (9) has a root ξ in the $GF[2^n]$ if and only if

$$\delta + \delta^2 + \ldots\ldots + \delta^{2^{n-1}} = 0.$$

The left member being its own square in the $GF[2^n]$ and hence either 0 or 1, it follows that (9) is irreducible in that field if and only if

$$\delta + \delta^2 + \delta^4 + \ldots\ldots + \delta^{2^{n-1}} = 1. \tag{10}$$

Applying to our quadratic form the transformation

$$\xi'_{m-1} = \xi_{m-1} + \lambda\xi_m, \quad \xi'_i = \xi_i, \qquad (i = 1, \ldots\ldots, m;\ i \neq m-1)$$

the constant δ is replaced by

$$\delta' \equiv \delta + \lambda + \lambda^2,$$

which is therefore a root of (10). Giving to λ all possible values in the $GF[2^n]$, we obtain the 2^{n-1} roots of (10). Indeed, if in the $GF[2^n]$,

$$\delta + \lambda + \lambda^2 = \delta + \lambda_1 + \lambda_1^2,$$

we must have $\lambda_1 = \lambda$ or $\lambda + 1$. Hence all irreducible quadratic forms in two variables of the $GF[2^n]$ can be transformed linearly into each other. For n odd, we can choose the form given by $\delta = 1$. Applying, finally, the transformation

$$\xi'_{m-1} = \delta^{\frac{1}{2}}\xi_{m-1}, \quad \xi'_m = \delta^{-\frac{1}{2}}\xi'_m, \quad \xi'_i = \xi_i \qquad (i = 1 \ldots\ldots m-2)$$

our form becomes $F_{\delta^{\frac{1}{2}}}$.

34. Changing the notation, we proceed to study the group G_λ of linear substitutions belonging to the $GF[2^n]$,

$$S: \begin{cases} \xi'_i = \displaystyle\sum_{j=1}^{m} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j), \\ \eta'_i = \displaystyle\sum_{j=1}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j) \end{cases} \qquad (i = 1, \ldots\ldots, m)$$

which leave absolutely invariant the function

$$F_\lambda \equiv \sum_{i=1}^{m} \xi_i\eta_i + \lambda\xi_1^2 + \lambda\eta_1^2.$$

The conditions on the coefficients are seen to be the following:

$$\left\{\begin{array}{ll} \sum_{i=1}^{m} (\alpha_{ij}\beta_{ik} + \alpha_{ik}\beta_{ij}) = 0, & \sum_{i=1}^{m} (\gamma_{ij}\delta_{ik} + \gamma_{ik}\delta_{ij}) = 0, \\ \sum_{i=1}^{m} (\alpha_{ij}\delta_{ik} + \gamma_{ik}\beta_{ij}) & = \begin{matrix} 0\ (j \neq k) \\ 1\ (j = k) \end{matrix}; \end{array}\right. \tag{11}$$

$$\left\{\begin{array}{l} \sum_{i=1}^{m} \alpha_{ij}\beta_{ij} + \lambda\alpha_{1j}^2 + \lambda\beta_{1j}^2 = \begin{matrix} 0\ (j > 1) \\ \lambda\ (j = 1) \end{matrix}, \\ \sum_{i=1}^{m} \gamma_{ij}\delta_{ij} + \lambda\gamma_{1j}^2 + \lambda\delta_{1j}^2 = \begin{matrix} 0\ (j > 1) \\ \lambda\ (j = 1) \end{matrix}. \end{array}\right. \tag{12}$$

It follows from the conditions (11) that S is an Abelian substitution on $2m$ indices in the $GF[2^n]$ and that its reciprocal is obtained by replacing α_{ij}, β_{ij}, γ_{ij}, δ_{ij} by respectively δ_{ji}, β_{ji} γ_{ji}, α_{ji}. By making this replacement in the relations (11) and (12), we obtain an equivalent set of relations (11_r) and (12_r)

35. Among the simplest substitutions leaving F_λ invariant occur the following [only the indices altered being written]:

$$\begin{array}{llll} N_{i,j,\kappa}: & \xi_i' = \xi_i + \kappa\eta_j, & \xi_j' = \xi_j + \kappa\eta_i, \\ R_{i,j,\kappa}: & \eta_i' = \eta_i' + \kappa\xi_j, & \eta_j' = \eta_j + \kappa\xi_i, \\ Q_{i,j,\kappa}: & \xi_i' = \xi_i + \kappa\xi_j, & \eta_j' = \eta_j + \kappa\eta_i, \\ T_{i,\kappa}\ : & \xi_i' = \kappa\xi_i \quad , & \eta_i' = \kappa^{-1}\eta_i \quad , \end{array}$$

where $i, j > 1$, if $\lambda \neq 0$;

$$\begin{array}{lll} N_{1,j,\kappa}: & \xi_1' = \xi_1 + \kappa\eta_j, & \xi_j' = \xi_j + \kappa\eta_1 + \lambda\kappa^2\eta_j, \\ R_{1,j,\kappa}: & \eta_1' = \eta_1 + \kappa\xi_j, & \eta_j' = \eta_j + \kappa\xi_1 + \lambda\kappa^2\xi_j, \\ Q_{1,j,\kappa}: & \xi_1' = \xi_1 + \kappa\xi_j, & \eta_j' = \eta_j + \kappa\eta_1 + \lambda\kappa^2\xi_j, \\ Q_{j,1,\kappa}: & \eta_1' = \eta_1 + \kappa\eta_j, & \xi_j' = \xi_j + \kappa\xi_1 + \lambda\kappa^2\eta_j, \end{array}$$

which, for $\lambda = 0$, fall under the above types;

$$M_i \equiv (\xi_i \eta_i) \quad , \quad P_{ij} \equiv (\xi_i \xi_j)(\eta_i \eta_j),$$
$$L: \quad \xi_1' = \eta_1, \quad \eta_1' = \xi_1 + \lambda^{-1}\eta_1,$$

where P_{1j} occurs in G only when $\lambda = 0$.

$$O_1^{\alpha,\,\delta}: \begin{cases} \xi_1' = \alpha\xi_1 + \lambda(\alpha+\delta)\eta_1, \\ \eta_1' = \lambda(\alpha+\delta)\xi_1 + \delta\eta_1, \end{cases} \quad [\alpha\delta + \lambda^2(\alpha^2+\delta^2) = 1].$$

36. For $\lambda = 0$ our group is the generalized first hypoabelian group G_0; for $\lambda = \lambda'$, where $\xi_1\eta_1 + \lambda'\xi_1^2 + \lambda'\eta_1^2$ is irreducible in the $GF[2^n]$, it is the generalized second hypoabelian group $G_{\lambda'}$. For $n = 1$, the structure of these groups was given by Jordan. The simplifications and corrections introduced by the writer* have been employed in the present paper. As far as practicable we treat together the groups G_0 and $G_{\lambda'}$. We do not completely determine the structure of G_0, that having been done in the paper cited and in more detail in a paper communicated November 10th, 1898, to the London Mathematical Society.

37. Theorem: *The groups G_0 and $G_{\lambda'}$ may be generated as follows:*

$$G_0 \equiv \{M_i,\ N_{i,j,\kappa}\}, \quad G_{\lambda'} \equiv \{M_i,\ N_{i,j,\kappa},\ O_1^{\alpha,\,\delta}\},$$

where $i, j = 1, 2, \ldots, m$, and κ is an arbitrary mark in the $GF[2^n]$.

We note that M_i transforms $N_{i,j,\kappa}$ into $Q_{j,i,\kappa}$ and $Q_{i,j,\kappa}$ into $R_{i,j,\kappa}$. Further, for $i, j > 1$ when $\lambda \neq 0$, we have

$$P_{ij} \equiv Q_{j,i,1}^{-1}\, Q_{i,j,1}\, Q_{j,i,1},$$
$$T_{i,\mu}\, T_{j,\mu} = M_i M_j\, P_{ij}\, R_{i,j,\mu^{-1}}\, N_{i,j,\mu}\, R_{i,j,\mu^{-1}}.$$

But M transforms $T_{j,\mu}$ into $T_{j,\mu^{-1}}$. Hence the group contains

$$T_{i,\mu}\, T_{j,\mu} \cdot T_{i,\mu}\, T_{j,\mu^{-1}} = T_{i,\mu^2}.$$

For the case $m = 2$, $\lambda = \lambda'$, the group $G_{\lambda'}$ contains

$$N_{1,2,\kappa}\, Q_{1,2,\kappa^{-1}\lambda^{-1}}\, N_{1,2,\kappa} = L M_1 M_2\, T_{2,\lambda\kappa^2}; \tag{13}$$

and therefore, since $L \equiv O_1^{0,\,\lambda^{-1}}$, it contains every $T_{2,\rho}$.

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* "The Structure of the Hypoabelian Groups," Bulletin of the American Mathematical Society, July, 1898.

To prove that every substitution S satisfying the relations (11) and (12) can be generated from the above substitutions, we first set up a substitution T derived from them which, like S, replaces ξ_m by

$$f \equiv \sum_{j=1}^{m} (\alpha_{mj}\xi_j + \gamma_{mj}\eta_j),$$

where, by (12_r),

$$\sum_{j=1}^{m} \alpha_{mj}\gamma_{mj} + \lambda\alpha_{m1}^2 + \lambda\gamma_{m1}^2 = 0. \tag{14}$$

a). If $\alpha_{mm} \neq 0$, we may take as T the product

$$T_{m\alpha_{mm}} \prod_{i=1}^{m-1} Q_{m,\,i,\,\alpha_{mi}} N_{i,\,m,\,\gamma_{mi}},$$

since it replaces ξ_m by

$$\sum_{j=1}^{m-1}(\alpha_{mj}\xi_j + \gamma_{mj}\eta_j) + \alpha_{mm}\xi_m + \alpha_{mm}^{-1}\left(\sum_{j=1}^{m-1}\alpha_{mj}\gamma_{mj} + \lambda\alpha_{m1}^2 + \lambda\gamma_{m1}^2\right)\eta_m,$$

which, by using (14), is seen to be f.

b). If $\alpha_{mm} = 0$, $\gamma_{mm} \neq 0$, we may take as T the product

$$T_{m\gamma_{mm}^{-1}} \prod_{i=1}^{m-1} Q_{i,\,m,\,\gamma_{mi}} R_{i,\,m,\,\alpha_{mi}} \cdot M_1 M_m.$$

c). If $\alpha_{mj} = \gamma_{mj} = 0$ $(j = m, m-1, \ldots, k-1)$, but α_{mk} and γ_{mk} not both zero, where $k > 1$, we may obtain, by case (a) or (b), a substitution T' replacing ξ_k by f and derived from the above generators. We may therefore take $T = T'P_{mk}$.

d). If $\alpha_{mj} = \gamma_{mj} = 0$ $(j = m, m-1, \ldots, 2)$, the proof given in (c) applies if $\lambda = 0$, so that P_{m1} belongs to the group. For $\lambda = \lambda'$, this case cannot exist, since the equation

$$\alpha_{m1}\gamma_{m1} + \lambda'\alpha_{m1}^2 + \lambda'\gamma_{m1}^2 = 0$$

requires $\alpha_{m1} = \gamma_{m1} = 0$ (whence $f \equiv 0$) on account of the irreducibility in the $GF[2^n]$ of the form $\xi_1\eta_1 + \lambda'\xi_1^2 + \lambda'\eta_1^2$.

It follows that $S = TS_1$, where S_1 leaves ξ_m fixed. Let S_1 replace η_m by

$$f' \equiv \sum_{j=1}^{m} (\beta_{mj}\xi_j + \delta_{mj}\eta_j).$$

Then by (11_r) we have $\delta_{mm} = 1$. Also by (12_r) we have

$$\sum_{j=1}^{m} \beta_{mj}\delta_{mj} + \lambda\delta_{m1}^2 + \lambda\beta_{m1}^2 = 0. \tag{15}$$

Then the product

$$S' \equiv \prod_{i=1}^{m-1} R_{i,\,m,\,\beta_{m1}}\, Q_{i,\,m,\,\delta_{m1}}$$

replaces ξ_m by ξ_m and η_m by

$$\sum_{j=1}^{m-1} (\beta_{mj}\xi_j + \delta_{mj}\eta_j) + \eta_m + \Big(\sum_{j=1}^{m-1} \beta_{mj}\delta_{mj} + \lambda\delta_{m1}^2 + \lambda\beta_{m1}^2\Big)\xi_m,$$

which equals f' since the coefficient of ξ_m is β_{mm} by (15).

We may therefore set $S_1 = S'S_2$, where S_2 leaves ξ_m and η_m fixed. It follows from the relations (11_r) that

$$\alpha_{im} = \beta_{im} = \gamma_{im} = \delta_{im} = 0. \qquad (i = 1, \ldots, m-1)$$

The relations holding between the α_{ij}, β_{ij}, γ_{ij}, δ_{ij} ($i, j = 1, \ldots, m-1$) are seen to be the relations (11) and (12) written for $m-1$ in place of m. Proceeding with S_2 as we did with S, etc., we find ultimately the result that $S = T'\Sigma$ where T' is derived from the above generators and Σ is a substitution of the group which affects ξ_1 and η_1 only.

38. We next determine the number and nature of the substitutions

$$\Sigma: \quad \xi_1' = \alpha\xi_1 + \gamma\eta_1, \quad \eta_1' = \beta\xi_1 + \delta\eta_1$$

which leave invariant $\xi_1\eta_1 + \lambda\xi_1^2 + \lambda\eta_1^2$. The conditions (11) and (12) become for the present case ($m = 1$):

$$\alpha\delta + \beta\gamma = 1, \quad \alpha\beta + \lambda\alpha^2 + \lambda\beta^2 = \lambda, \quad \gamma\delta + \lambda\gamma^2 + \lambda\delta^2 = \lambda. \tag{16}$$

Expressing the same conditions for the reciprocal of Σ, we get,

$$\delta\beta + \lambda\delta^2 + \lambda\beta^2 = \lambda, \quad \gamma\alpha + \lambda\gamma^2 + \lambda\alpha^2 = \lambda. \tag{17}$$

Combining (17) with the last two of (16), we find

$$\beta(\alpha+\delta)=\gamma(\alpha+\delta)=\lambda(\alpha+\delta)^2, \tag{18}$$

which may be taken to replace (17).

a). Suppose that $\alpha \neq \delta$. Then by (18)

$$\beta=\gamma=\lambda(\alpha+\delta), \tag{18'}$$

when the conditions (16) all reduce to

$$\alpha\delta+\lambda^2\alpha^2+\lambda^2\delta^2=1. \tag{19}$$

If $\lambda=0$, the substitution Σ becomes $T_{1,\alpha}$. If, however, $\lambda=\lambda'$, so that $\xi_1\xi_2+\lambda\xi_1^2+\lambda\xi_2^2$ is irreducible, the only set of solutions in the $GF[2^n]$ of $\alpha\delta+\lambda^2\alpha^2+\lambda^2\delta^2=0$ is $\alpha=\delta=0$. Each one of the remaining $2^{2n}-1$ sets of values α_1, δ_1 in the $GF[2^n]$ make

$$\alpha_1\delta_1+\lambda^2\alpha_1^2+\lambda^2\delta_1^2=x^2\neq 0.$$

Then will α_1/x, δ_1/x be a set of solutions of (19) and inversely. Hence the number of distinct sets of solutions* of (19) is

$$(2^{2n}-1)/(2^n-1)=2^n+1.$$

b). Suppose next that $\alpha=\delta$, so that the conditions (18) become identities. From the last two of (16) we find that

$$\alpha(\beta+\gamma)=\lambda(\beta+\gamma)^2.$$

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*If n be odd, we may take $\lambda=1$. Among the solutions occur

$$(\alpha,\delta)=(0,1),(1,0),(1,1).$$

For $n=1$, there are no other solutions. For $n=3$, we find also

$$(\alpha,\delta)=(\rho,\rho^2),\quad(\rho,\rho^4),\quad(\rho^2,\rho),\quad(\rho^2,\rho^4),\quad(\rho^4,\rho),\quad(\rho^4,\rho^2)$$

where ρ is a definite root of the congruence $\rho^3=\rho+1$, irreducible modulo 2. For $n=5$, we derive from (19)

$$\alpha^{32}=\alpha\delta^{31}+\delta^{32}+\delta^{30}+\delta^{28}+\delta^{24}+\delta^{16}+1=0.$$

But $\delta^{32}+\delta^{30}+\delta^{28}+\delta^{24}+\delta^{16}+1=(\delta+1)^2(\delta^5+\delta^3+\delta^2+\delta+1)^2(\delta^5+\delta^4+\delta^3+\delta+1)^2(\delta^5+\delta^4+\delta^2+\delta+1)^2$. These three quintics irreducible modulo 2 furnish 2.5.3 sets of solutions, which with the above three give 2^5+1 sets.

If $\beta = \gamma$, we find from (16)

$$\alpha^2 + \beta^2 = 1,\ \alpha\beta = 0.$$

Hence Σ is either the identity or $M_1 \equiv (\xi_1 \eta_1)$.

If $\beta \neq \gamma$, then $\alpha = \lambda(\beta + \gamma)$ and all the relations (16) reduce to

$$\beta\gamma + \lambda^2\beta^2 + \lambda^2\gamma^2 = 1.$$

By interchanging α with γ and β with δ, the present relations take the form (18′) and (19), which lead to the substitution Σ_1, we will say. Hence the present substitution Σ is the product $M_1\Sigma_1$. The total number of substitutions leaving $\xi_1\eta_1 + \lambda\xi_1^2 + \lambda\eta_1^2$ invariant is therefore $2(2^n + 1)$, if the form be irreducible, and $2(2^n - 1)$ if it be reducible in the $GF[2^n]$.

39. We can now readily determine the order $\Omega^{(\lambda)}_{m,n}$ of G^λ, including the cases $\lambda = 0$ and $\lambda = \lambda'$. The number of distinct linear functions f by which the substitutions of G_λ can replace ξ_m is $P^{(\lambda)}_{m,n} - 1$, if $P^{(\lambda)}_{m,n}$ denote the number of sets of solutions in the $GF[2^n]$ of the equation (14). For $m > 1$, the pair of equations

$$\alpha_{mm}\gamma_{mm} = \tau, \quad \sum_{j=1}^{m-1} \alpha_{mj}\gamma_{mj} + \lambda\alpha_{m1}^2 + \lambda\gamma_{m1}^2 = \tau$$

has $(2^{n+1} - 1)\,P^{(\lambda)}_{m-1,n}$ sets of solutions when $\tau = 0$ and $(2^n - 1)(2^{n(2m-2)} - P^{(\lambda)}_{m-1,n})$ sets of solutions when τ runs through the marks $\neq 0$ of the $GF[2^n]$. Hence we have the recursion formula,

$$P^{(\lambda)}_{m,n} = 2^n P^{(\lambda)}_{m-1,n} + (2^n - 1)\,2^{n(2m-2)}. \tag{20}$$

For $\lambda = 0$, $P^{(0)}_{1,n} = 2(2^n - 1)$ and we find by induction that

$$P^{(0)}_{s,n} - 1 = (2^{ns} - 1)(2^{n(s-1)} + 1).$$

For $\lambda = \lambda'$, $P^{(\lambda')}_{1,n} = 1$, since $\alpha = \gamma = 0$ is the only set of solutions in the $GF[2^n]$ of $\alpha\gamma + \lambda'\alpha^2 + \lambda'\gamma^2 = 0$. We prove by induction that

$$P^{(\lambda')}_{s,n} - 1 = (2^{ns} + 1)(2^{n(s-1)} - 1).$$

The number of distinct linear functions f' is $2^{n(2m-2)}$. Indeed, since $\delta_{mm} = 1$, the relation (15) determines β_{mm} in terms of β_{mj}, δ_{mj} $(j = 1, \ldots, m-1)$, which may be chosen arbitrarily in the $GF[2^n]$. It follows, therefore, from §37, that

$$\Omega^{(\lambda)}_{m,n} = (P^{(\lambda)}_{m,n} - 1)\,2^{2n(m-1)}\Omega^{(\lambda)}_{m-1,n}.$$

But, by §38, we have the initial values

$$\Omega^{(0)}_{1,n} = 2(2^n - 1), \quad \Omega^{(\lambda')}_{1,n} = 2(2^n + 1).$$

We now readily obtain the formulæ

$$\Omega^{(0)}_{m,n} = (2^{nm} - 1)[(2^{2n(m-1)} - 1)\, 2^{2n(m-1)}][(2^{2n(m-2)} - 1)\, 2^{2n(m-2)}] \ldots [(2^{2n} - 1)\, 2^{2n}]\, 2,$$
$$\Omega^{(\lambda')}_{m,n} = (2^{nm} + 1)[(2^{2n(m-1)} - 1)\, 2^{2n(m-1)}] \ldots [(2^{2n} - 1)\, 2^{2n}]\, 2.$$

40. In determining the structure of G_λ, we shall find that there exists a subgroup J_λ characterized by the additional relation between the coefficients

$$I(\alpha, \beta, \gamma, \delta) \equiv \sum_{i,j}^{1 \ldots m} \alpha_{ij}\delta_{ij} + \lambda^2 (\alpha_{11}^2 + \beta_{11}^2 + \gamma_{11}^2 + \delta_{11}^2) = m. \qquad (21)$$

We shall prove that all the substitutions of G_λ which satisfy (21) form a group and that this group can be generated as follows:

$$J_0 \equiv \{M_iM_j,\ N_{i,j,\kappa}\}, \quad J_{\lambda'} \equiv \{M_iM_j,\ N_{i,j,\kappa},\ O_1^{\alpha,\delta}\},$$

new generators, as $T_{1,\kappa}$ and $Q_{1,2,\kappa}$ being necessary in J_0 if $m = 2$ [see note to §50].

We first prove that every substitution of the group J_λ satisfies the relation (21). It is evidently satisfied by the generators; for example, for $O_1^{\alpha,\delta}$ we find

$$I(\alpha, \beta, \gamma, \delta) = (m - 1) + \alpha\delta + \lambda^2(\alpha^2 + \delta^2) \equiv m \pmod{2}.$$

To give a proof by induction, we suppose that a substitution Σ satisfies (21) and prove that the products $M_iM_j\Sigma$, $N_{i,j,\kappa}\Sigma$, $O_1^{\alpha,\delta}\Sigma$ will satisfy (21), whereas the product $M_j\Sigma$ will not.

a). The coefficients $\bar{\alpha}_{ij}, \bar{\beta}_{ij}, \ldots,$ of $M_j\Sigma$ are as follows:

$$\bar{\alpha}_{ij} = \gamma_{ij}, \quad \bar{\gamma}_{ij} = \alpha_{ij}, \quad \bar{\beta}_{ij} = \delta_{ij}, \quad \bar{\delta}_{ij} = \beta_{ij}, \qquad (i = 1, \ldots, m)$$
$$\bar{\alpha}_{ik} = \alpha_{ik}, \quad \bar{\beta}_{ik} = \beta_{ik}, \quad \bar{\gamma}_{ik} = \gamma_{ik}, \quad \bar{\delta}_{ik} = \delta_{ik}. \quad \begin{pmatrix} i = 1, \ldots, m \\ k = 1, \ldots, m; k \neq j \end{pmatrix}$$

Hence

$$I(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}) = \sum_{k \neq j}^{i,k=1 \ldots m} \alpha_{ik}\delta_{ik} + \sum_{i=1}^{m} \gamma_{ij}\beta_{ij} + \lambda^2(\alpha_{11}^2 + \beta_{11}^2 + \gamma_{11}^2 + \delta_{11}^2)$$
$$= I(\alpha, \beta, \gamma, \delta) + \sum_{i=1}^{m} (\gamma_{ij}\beta_{ij} - \alpha_{ij}\delta_{ij}) = m + 1.$$

Hence $M_j\Sigma$ does not satisfy (21), while $M_iM_j\Sigma$ does.

b). The coefficients $\overline{\alpha}_{ij}$, etc., of $N_{1,j,\kappa}\Sigma$ are as follows:

$$\begin{aligned}
&\overline{\alpha}_{rs} = \alpha_{rs}, \quad \overline{\beta}_{rs} = \beta_{rs}, && (r, s = 1, \ldots, m)\\
&\overline{\gamma}_{rs} = \gamma_{rs}, \quad \overline{\delta}_{rs} = \delta_{rs}, && (r, s = 1, \ldots, m;\ s \neq 1, j)\\
&\overline{\gamma}_{r1} = \gamma_{r1} + \kappa\alpha_{rj}, \quad \overline{\gamma}_{rj} = \gamma_{rj} + \kappa\alpha_{r1} + \lambda\kappa^2\alpha_{rj}, && (r = 1, \ldots, m)\\
&\overline{\delta}_{r1} = \delta_{r1} + \kappa\beta_{rj}, \quad \overline{\delta}_{rj} = \delta_{rj} + \kappa\beta_{r1} + \lambda\kappa^2\beta_{rj}. && (r = 1, \ldots, m)
\end{aligned}$$

Hence $I(\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta})$ equals

$$\begin{aligned}
&\sum_{s \neq 1, j}^{r, s = 1 \ldots m} \alpha_{rs}\delta_{rs} + \sum_{r=1}^{m} \alpha_{r1}(\delta_{r1} + \kappa\beta_{rj}) + \sum_{r=1}^{m} \alpha_{rj}(\delta_{rj} + \kappa\beta_{r1} + \lambda\kappa^2\beta_{rj})\\
&\qquad\qquad + \lambda^2\{\alpha_{11}^2 + \beta_{11}^2 + (\gamma_{11} + \kappa\alpha_{1j})^2 + (\delta_{11} + \kappa\beta_{1j})^2\}\\
&= \sum_{r, s}^{1 \ldots m} \alpha_{rs}\delta_{rs} + \lambda^2(\alpha_{11}^2 + \beta_{11}^2 + \gamma_{11}^2 + \delta_{11}^2) + \kappa\sum_{r=1}^{m}(\alpha_{r1}\beta_{rj} + \alpha_{rj}\beta_{r1})\\
&\qquad\qquad + \lambda\kappa^2\left(\sum_{r=1}^{m}\alpha_{rj}\beta_{rj} + \lambda\alpha_{1j}^2 + \lambda\beta_{1j}^2\right),
\end{aligned}$$

which equals $I(\alpha, \beta, \gamma, \delta)$ since the last two sums are zero by (11) and (12).

An analogous proof holds for the products $N_{i,j,\kappa}\Sigma$ $(i, j > 1)$.

c). The coefficients in the product $O_1^{\alpha,\delta}\Sigma$ are

$$\begin{aligned}
&\overline{\alpha}_{ij} = \alpha_{ij}, \quad \overline{\beta}_{ij} = \beta_{ij}, \quad \overline{\gamma}_{ij} = \gamma_{ij}, \quad \overline{\delta}_{ij} = \delta_{ij}, && (i, j = 2, \ldots, m)\\
&\overline{\alpha}_{i1} = \alpha\alpha_{i1} + \lambda(\alpha + \delta)\gamma_{i1}, \quad \overline{\gamma}_{i1} = \lambda(\alpha + \delta)\alpha_{i1} + \delta\gamma_{i1}, && \\
&\overline{\beta}_{i1} = \alpha\beta_{i1} + \lambda(\alpha + \delta)\delta_{i1}, \quad \overline{\delta}_{i1} = \lambda(\alpha + \delta)\beta_{i1} + \delta\delta_{i1}. && (i = 1, \ldots, m)
\end{aligned}$$

Using (11), (12) and (19), we may verify that

$$I(\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}) = I(\alpha, \beta, \gamma, \delta).$$

d). It follows from the remarks at the beginning of §37 that the substitutions $Q_{i,j,\kappa}$, $R_{i,j,\kappa}$ $(i, j = 1, \ldots, m)$ and $P_{i,j}$, $T_{i,\kappa}$ $(i, j > 1$ if $\lambda = \lambda')$ satisfy the relation (21) and likewise their products by Σ.

Inversely, every substitution S *satisfying the relations* (11), (12) and (21) *belongs to the group* J_λ.

For $m > 2$, the group J_λ contains $Q_{i,j,\kappa}$, the transformed of $N_{i,j,\kappa}$ by M_jM_k $(k \neq i, j)$; also $R_{i,j,\kappa}$ and $Q_{j,i,\kappa}$, the transformed of $N_{i,j,\kappa}$ and $Q_{i,j,\kappa}$

respectively by $M_i M_j$. Then by §37, it contains P_{ij}, $T_{i,\kappa} T_{j,\kappa}$ $(i, j > 1)$ and $T_{i,\kappa} T_{j,\kappa^{-1}}$, the transformed of the latter by $M_1 M_j$. The product of the two gives T_{i,κ^2}.

For $m = 2$, $\lambda = \lambda'$, the group $J_{\lambda'}$ contains

$$Q_{2,1,\kappa} = L^{-1} N_{1,2,\kappa} L,$$

and therefore $R_{1,2,\kappa}$ and $Q_{1,2,\kappa}$, the transformed of $N_{1,2,\kappa}$ and $Q_{2,1,\kappa}$ respectively by $M_1 M_2$. It thus contains $T_{2,\rho}$ by (13).

By the proof in §§37–38, every substitution of G_λ is of one of the two forms K or KM_1, where K is derived from the $M_i M_j$, $N_{i,j,\kappa}$, $Q_{i,j,\kappa}$, $R_{i,j,\kappa}$ $(i, j = 1, \ldots, m)$; $O_1^{\alpha,\delta}$, $T_{i,\kappa}$, P_{ij} $(i, j > 1)$. We may therefore state the theorem:

The group G_λ *contains a subgroup* J_λ *of index* 2, *which* M_1 *extends to the total group* G_λ.

41. Theorem: *The Group* $J_{\lambda'}$ *may be generated by the substitutions*

$$L,\ M_i M_j,\ N_{i,j,\kappa}. \qquad (i, j = 1, \ldots, m)$$

As it does not readily appear that every $O_1^{\alpha,\delta}$ can be expressed in terms of the above substitutions [which fact is the gist of our theorem], we give a direct proof of the theorem. In contrast to the method of §37, we begin here by considering the indices ξ_1, η_1 which play a special rôle in our group $J_{\lambda'}$. We shall obtain certain results needed in §43.

Let any given substitution S of $J_{\lambda'}$ replace ξ_1 by

$$\sum_{j=1}^{m} (\alpha_{1j}\xi_j + \gamma_{1j}\eta_j),$$

where by (12_r)

$$\sum_{j=1}^{m} \alpha_{1j}\gamma_{1j} + \lambda\alpha_{11}^2 + \lambda\gamma_{11}^2 = \lambda. \tag{22}$$

If α_{1j}, γ_{1j} $(j = 2, \ldots, m)$ are all zero, $Q_{2,1,1}\, Q_{1,2,1}\, S$ will replace ξ_1 by

$$\gamma_{11}\eta_1 + \alpha_{11}\xi_2 + (\gamma_{11} + \lambda\alpha_{11})\,\eta_2,$$

in which α_{11} and $\gamma_{11} + \lambda\alpha_{11}$ are not both zero by (22). We may therefore confine ourselves to substitutions S in which not every α_{1j}, γ_{1j} $(j = 2, \ldots, m)$ is zero, and in particular may assume that $\alpha_{12} \neq 0$.

The product $N_{1,2,\kappa}\,S$ replaces ξ_1 by

$$\alpha_{11}\xi_1 + (\gamma_{11} + \kappa\alpha_{12})\,\eta_1 + \alpha_{12}\xi_2 + (\gamma_{12} + \kappa\alpha_{11} + \lambda\kappa^2\alpha_{12})\,\eta_2 + \dots .$$

We may, by choice of κ, make the coefficient of η_1 zero. Then in $S' \equiv LN_{1,2,\kappa}S$, we have $\alpha_{11} = 0$, $\alpha_{12} \neq 0$. As before, the product $N_{1,2,\mu}S' \equiv S''$ will replace ξ_1 by

$$(\gamma_{11} + \mu\alpha_{12})\,\eta_1 + \alpha_{12}\xi_2 + \dots .$$

By determining μ, we can make the coefficient of η_1 unity. The substitution S'' therefore has

$$\alpha_{11} = 0, \quad \gamma_{11} = 1, \quad \sum_{j=2}^{m} \alpha_{1j}\gamma_{1j} = 0, \quad \alpha_{12} \neq 0.$$

It follows, by §37, that there exists a substitution T, derived from

$$M_iM_j, \quad N_{i,j,\kappa}, \quad T_{i,\kappa}, \quad Q_{i,j,\kappa}, \qquad (i, j = 2, \dots, m) \tag{23}$$

which replaces η_2 by $\sum_{j=2}^{m} (\alpha_{1j}\xi_j + \gamma_{1j}\eta_j)$. Hence the product

$$S_1 \equiv M_1M_2Q_{2,1,1}T^{-1}S''$$

will leave ξ_1 fixed.

It follows that the given substitution $S = \Sigma S_1$, where Σ is derived from L, M_iM_j, $N_{i,j,\kappa}$. Let S_1 replace η_1 by

$$\sum_{j=1}^{m} (\beta_{1j}\xi_j + \delta_{1j}\eta_j),$$

where by (11_r) and (12_r)

$$\delta_{11} = 1, \quad \sum_{j=1}^{m} \beta_{1j}\delta_{1j} + \lambda\beta_{11}^2 = 0 \cdot \tag{24}$$

If $\beta_{1j} = \delta_{1j} = 0$ $(j = 2, \dots, m)$, then $\beta_{11} = 0$ or λ^{-1}. Hence S_1 or $L^{-1}M_1M_2S_1$ respectively will leave ξ_1 and η_1 fixed.

If $\beta_{12} \neq 0$, for example, then $Q_{2,1,\kappa}S_1$ leaves ξ_1 fixed and replaces η_1 by

$$\eta_1 + (\beta_{11} + \kappa\beta_{12})\,\xi_1 + \beta_{12}\xi_2 + (\delta_{12} + \kappa + \lambda\kappa^2\beta_{12})\,\eta_2 + \dots .$$

By choice of κ we may make the coefficient of ξ_1 zero. In the resulting substitution S_1', we have

$$\beta_{11} = 0, \quad \delta_{11} = 1, \quad \sum_{j=2}^{m} \beta_{1j}\,\delta_{1j} = 0, \quad \beta_{12} \neq 0.$$

As above, there exists a substitution T' in $J_{\lambda'}$ which replaces η_2 by

$$\sum_{j=2}^{m} (\beta_{1j}\xi_j + \delta_{1j}\eta_j)$$

without altering ξ_1 and η_1. Then will $S_2 \equiv Q_{2,1,1} T^{-1} S_1'$ leave ξ_1 and η_1 fixed. But, by §37, the substitution S_2 affecting only ξ_i, η_i $(i = 2, \ldots, m)$ can be derived from the substitutions (23).

42. We can make a new determination of the order of $J_{\lambda'}$. The number of sets of solutions of (22) is

$$(2^{2nm} - P^{(\lambda')}_{m,n})/(2^n - 1) \equiv (2^{nm} + 1)\, 2^{n(m-1)},$$

where $P^{(\lambda')}_{m,n} \equiv 2^{n(2m-1)} - 2^{nm} + 2^{n(m-1)}$ is the number of sets of solutions of

$$\sum_{j=1}^{m} \alpha_{1j}\gamma_{1j} + \lambda\alpha_{11}^2 + \lambda\gamma_{11}^2 = 0.$$

By a slight calculation we find that the number of sets of solutions of (24) is $(2^{n(m-1)} + 1)\, 2^{n(m-1)}$. Hence

$$\Omega^{(\lambda')}_{m,n} = (2^{nm} + 1)(2^{n(m-1)} + 1)\, 2^{2n(m-1)} \Omega^{(0)}_{m-1,n},$$

so that from the order of the first hypoabelian group we readily derive that of the second hypoabelian group.

Simplicity of the Group $J_{\lambda'}$, §§43 46.

43. Let I be an invariant subgroup of* $J_{\lambda'}$ containing a substitution S not the identity,

$$S: \begin{cases} \xi_i' = \sum_{j=1}^{m} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j), \\ \eta_i' = \sum_{j=1}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j), \end{cases} \qquad (i = 1, \ldots, m)$$

Proposition I.—*I contains a substitution, not the identity, which leaves ξ_1 fixed.*

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* In the following paragraphs the subscript λ' will be dropped from $J_{\lambda'}$.

a). If $\gamma_{11} \neq 0$, J contains a substitution T which leaves ξ_1 fixed and replaces η_1 by

$$\sum_{j=1}^{m} (\alpha_{1j}\xi_j + \gamma_{1j}\eta_j).$$

Hence I contains $T^{-1}ST \equiv S_1$ which replaces ξ_1 by η_1.

If S_1 leaves ξ_2, η_2, ξ_3, η_3 unaltered, I will contain its transformed by the following substitution belonging to J:

$$W: \begin{cases} \xi_1' = \xi_2 + \lambda\eta_2 , & \eta_1' = \eta_2 + \lambda\xi_3 + \eta_3 ; \\ \xi_2' = \xi_1 + \lambda\xi_3 + \eta_3 , & \eta_2' = \lambda\xi_1 + \eta_1 + \lambda^2\xi_3 + \lambda\eta_3 , \\ \xi_3' = \eta_1 + \xi_2 + \lambda\eta_2 + \xi_3 , & \eta_3' = \lambda\eta_1 + \lambda\xi_2 + \lambda^2\eta_2 + \eta_3 . \end{cases}$$

This transformed leaves ξ_1 and η_1 fixed.

In the contrary case, J contains a substitution T, leaving ξ_1 and η_1 fixed but not commutative with S_1; hence I contains $S_1^{-1}T^{-1}S_1T \neq 1$ which leaves ξ_1 fixed. Indeed, comparing the values by which $S_1R_{2,3,\kappa}$ and $R_{2,3,\kappa}S_1$ replace η_2, we must have

$$\xi_2' = (\;)\,\xi_2 + (\;)\,\xi_3,$$

if S_1 be commutative with $R_{2,3,\kappa}$. Comparing the values by which $S_1Q_{3,2,\kappa}$ and $Q_{3,2,\kappa}S_1$ replace ξ_3, we must have

$$\xi_2' = (\;)\,\xi_2 + (\;)\,\eta_3.$$

Hence $\xi_2' = \alpha\xi_2$. If S_1 be commutative with M_2M_3, we have also $\eta_2' = \alpha\eta_2$. Hence $\alpha^2 = 1$ or $\alpha = 1$. Lastly, if S_1 be commutative with P_{23}, we must have

$$\xi_3' = \xi_3, \quad \eta_3' = \eta_3.$$

There remains the case $m = 2$. If $n > 1$, there exists in the $GF[2^n]$ a mark $\kappa \neq 1, \neq 0$. If S be not commutative with $T_{2,\kappa}$, then $S_1^{-1}T_{2\kappa}^{-1}S_1T_{2\kappa}$ is a substitution belonging to I, leaving ξ_1 fixed and different from the identity. If, however, $S_1T_{2\kappa} = T_{2\kappa}S_1$, we readily find that S_1 must have the form

$$\xi_1' = \eta_1, \quad \eta_1' = \xi_1 + \delta_{11}\eta_1, \quad \xi_2' = \alpha\xi_2, \quad \eta_2' = \alpha^{-1}\eta_2.$$

The relation (21) gives $\delta_{11} = \lambda^{-1}$. Hence $S_1 = LT_{2\alpha}$. Since

$$N_{1,2,\kappa}^{-1}T_{2\alpha}N_{1,2,\kappa} = N_{1,2,\kappa+\kappa\alpha^{-1}}\,T_{2\alpha},$$
$$N_{1,2,\kappa}^{-1}LN_{1,2,\kappa} = LQ_{2,1,\kappa}N_{1,2,\kappa},$$

it follows that $N_{1,2,\kappa}$ transforms S_1 into

$$N_{1,2,\kappa+\kappa\alpha^{-1}}T_{2\alpha}LQ_{2,1,\kappa}N_{1,2,\kappa}.$$

Hence I contains

$$Q_{2,1,\kappa}N_{1,2,\kappa}N_{1,2,\kappa+\kappa\alpha^{-1}} \equiv Q_{2,1,\kappa}N_{1,2,\kappa\alpha^{-1}},$$

in which the coefficient of γ_{11} is zero.

b). $\gamma_{11}=0$. If $\alpha_{1j}=\gamma_{1j}=0$ $(j=2,\ldots\ldots,m)$, S leaves ξ_1 fixed. In the contrary case we may suppose that $\alpha_{13}\neq 0$, when $m \gtreqless 3$.

Transforming S by $N_{2,3,\kappa}$ we obtain a substitution S' which replaces ξ_1 by

$$\alpha_{11}\xi_1+\alpha_{12}\xi_2+\alpha_{13}\xi_3+(\gamma_{12}+\kappa\alpha_{13})\,\eta_2+(\gamma_{13}+\kappa\alpha_{12})\,\eta_3+\ldots\ldots$$

We may therefore make $\alpha_{12}=\gamma_{12}+\kappa\alpha_{13}$. Hence in S' we have $\alpha'_{12}=\gamma'_{12}$. Then I contains the substitution

$$S_1=S'^{-1}L^{-1}M_1M_2S'M_1M_2L$$

which leaves ξ_1 fixed. If S_1 reduce to the identity, we find, by comparing the expressions by which S' and $L^{-1}M_1M_2S'M_1M_2L$ replace η_1, that

$$\lambda^{-1}\xi'_1=\lambda^{-1}\delta'_{11}\xi_1+(\beta'_{12}+\delta'_{12})(\xi_2+\eta_2).$$

Then the transformed of S' by $N_{2,3,\kappa}$ will give a substitution $\overline{S}$ which replaces $\lambda^{-1}\xi_1$ by

$$\delta'_{11}\lambda^{-1}\xi_1+(\beta'_{12}+\delta'_{12})(\xi_2+\eta_2+\kappa\eta_3).$$

Using $\overline{S}$ in place of our given S', the product denoted by S_1 will not be the identity and will leave ξ_1 fixed.

For $m=2$, we may suppose that $\alpha_{12}\neq 0$. Transforming S by $Q_{2,1,\kappa}$ we obtain a substitution S' which replaces ξ_1 by

$$(\alpha_{11}+\kappa\alpha_{12})\,\xi_1+\alpha_{12}\xi_2+(\gamma_{12}+\lambda\kappa^2\alpha_{12})\,\eta_2.$$

We may therefore suppose that the coefficient of η_2 is zero. From (12_r) we get $\alpha_{11}=1$, since $\gamma_{12}=\gamma_{11}=0$. Transforming by $T_{2\kappa}$, we may suppose that $\alpha_{12}=1$. Hence we have a substitution S which replaces ξ_1 by $\xi_1+\xi_2$.

The group I therefore contains $S'\equiv S^{-1}R_{1,2,\kappa}^{-1}SR_{1,2,\kappa}$ which replaces ξ_1 by ξ_1. If it be the identity, we find by equating the values by which $SR_{1,2,\kappa}$ and $R_{1,2,\kappa}S$ replace η_1 that

$$\xi'_2=\delta_{12}\xi_1+(\delta_{11}+\lambda\kappa\delta_{12})\,\xi_2.$$

By (12_r) we have $\delta_{12}=0$; by (11_r), $\delta_{11}=1$. Hence S would be of the form

$$\xi_1'=\xi_1+\xi_2,\quad \eta_1'=\beta_{11}\xi_1+\eta_1+\beta_{12}\xi_2,\quad \xi_2'=\xi_2,\quad \eta_2'=\beta_{21}\xi_1+\eta_1+\beta_{22}\xi_2+\eta_2.$$

The reciprocal of S replaces ξ_1 by $\xi_1+\xi_2$ and may therefore be used in place of S. But S^{-1} is evidently commutative with $R_{1,2,\kappa}$ only if $\beta_{11}=0$. Then by (12_r) we have $\beta_{12}=\beta_{21}$. Hence

$$S=R_{1,2,\beta_{12}}Q_{1,2,1}.$$

This is transformed by L into

$$S_1\equiv R_{1,2,1+\lambda^{-1}\beta_{12}}Q_{1,2,\beta_{12}}.$$

Hence if $\beta_{12}=0$, I contains $R_{1,2,1}$ which leaves ξ_1 fixed. If $\beta_{12}\neq 0$, we transform S by $T_{2,\beta_{12}^{-1}}$ and obtain

$$S'=R_{1,2,\beta_{12}^2}Q_{1,2,\beta_{12}}$$

Hence I contains $S_1^{-1}S'=R_{1,2,\rho}$ where

$$\rho\equiv 1+\lambda^{-1}\beta+\beta^2\neq 0$$

since the form $\lambda\xi_1^2+\lambda\xi_2^2+\xi_1\xi_2$ is ireducible in the field.

44. Proposition II.—*If* $(m, n)\neq(2, 1)$, *the group* I *contains a substitution, not the identity, which leaves* ξ_1 *and* η_1 *fixed.*

We have proven that I contains a substitution S, leaving ξ_1 fixed. Let it replace η_1 by

$$\sum_{j=1}^{m}(\beta_{1j}\xi_j+\delta_{1j}\eta_j),$$

where

$$\delta_{11}=1,\quad \sum_{j=1}^{m}\beta_{1j}\delta_{1j}+\lambda\beta_{11}^2=0. \tag{24}$$

a). If $\beta_{1j}=\delta_{1j}=0$ $(j=2, \ldots, m)$, we proceed as in case (a) of the preceding paragraph. If S leaves ξ_2, η_2, ξ_3, η_3 unaltered, its transform by W will leave ξ_1 and η_1 fixed. In the contrary case J will contain a substitution T, leaving ξ_1 and η_1 fixed and not commutative with S. Hence I contains $S^{-1}T^{-1}ST\neq 1$ which leaves both ξ_1 and η_1 fixed, since S replaces ξ_1 and η_1 by functions of ξ_1 and η_1 only. For $m=2$, $n>1$, I contains a substitution $R_{1,2,\rho}\neq 1$ by the proof in the last paragraph. It therefore contains its transform by $T_{2,\sigma}$, giving

$R_{1,2,\rho\sigma-1}$, and hence contains every $N_{1,2,\kappa}$. Therefore I contains every $Q_{1,2,\kappa}$ and, by (13), every $LM_1M_2T_{2,\lambda\kappa^2}$, and finally, every $T_{2,\rho}$, viz.

$$(LM_1M_2T_{2,\lambda})^{-1}(LM_1M_2T_{2,\lambda\kappa^2}) = T_{2,\lambda\kappa^2+\lambda-1}.$$

The substitution $T_{2,\rho} \neq 1$ leaves ξ_1 and η_1 fixed.

b). If $\beta_{12} \neq 0$, for example, the transformed of S by $T_{2,\beta_{12}}$ gives a substitution S' in which $\beta_{12} = 1$. By §37, J contains a substitution T, leaving ξ_1 and η_1 fixed and replacing ξ_2 by

$$\xi_2 + \tau\eta_2 + \sum_{j=3}^{m}(\beta_{1j}\xi_j + \delta_{1j}\eta_j),$$

the exact value of τ being immaterial here. Then I contains $S_1 \equiv T^{-1}S'T$ which replaces ξ_1 by ξ_1 and η_1 by

$$\beta_{11}'\xi_1 + \eta_1 + \beta_{12}'\eta_2 + \xi_2.$$

b$_1$). If $\beta_{12}' \neq 0$, the transformed S_2 of S_1 by $T_{2,\mu}^{-1}$ will replace η_1 by

$$\beta_{11}'\xi_1 + \eta_1 + \mu(\xi_2 + \eta_2),$$

if we take $\mu = \beta_{12}'^{2}$. Let V be any substitution of J which leaves ξ_1, η_1 and $\xi_2 + \eta_2$ fixed. Then $S_3 \equiv S_2^{-1}V^{-1}S_2V$ belongs to I and leaves ξ_1 and η_1 fixed. There remains the case in which S_2 is commutative with every V. If S_2 be commutative with $V = Q_{3,2,\kappa}N_{2,3,\kappa}$, we find, on comparing the two values by which the products S_2V and VS_2 replace ξ_2, that

$$\eta_3' = \alpha_{23}'(\xi_2 + \eta_2) + (\alpha_{22}' + \gamma_{22}' + \kappa\alpha_{23}')\eta_3.$$

Then, by (12$_r$), $\alpha_{23}' = 0$, so that $\eta_3' = \delta_{33}'\eta_3$. Taking $V = M_2M_3$, it follows that $\xi_3' = \delta_{33}'\xi_3$. Hence, by (11$_r$), $\delta_{33}'^2 = 1$, so that S_2 leaves ξ_3, η_3 fixed. If $m > 3$, by taking $V = P_{3,i}$, we see that we can suppose that S_2 leaves ξ_i, η_i $(i = 3, \ldots, m)$ fixed. Since S_2 is commutative with M_2M_3, it has the form

$$S_2:\ \begin{cases} \xi_1' = \xi_1, \quad \eta_1' = \beta_{11}'\xi_1 + \eta_1 + \mu(\xi_2 + \eta_2), \\ \xi_2' = \alpha_{21}'\xi_1 + \alpha_{22}'\xi_2 + \gamma_{22}'\eta_2, \quad \eta_2' = \alpha_{21}'\xi_1 + \gamma_{22}'\xi_2 + \alpha_{22}'\eta_2. \end{cases}$$

Hence, by (11), $\alpha_{22}'^2 + \gamma_{22}'^2 = 1$ or $\alpha_{22}' + \gamma_{22}' = 1$, a result found above. By (11) and (21) we find, respectively,

$$\alpha_{21}' = \mu(\alpha_{22} + \gamma_{22}) = \mu, \quad \lambda\beta_{11}' = \alpha_{22}' + 1 = \gamma_{22}'.$$

The transformed of S_2 by $R_{1,2,\kappa}$ gives a substitution which leaves ξ_1 fixed and replaces η_1 by

$$(\beta'_{11} + \kappa^2\gamma'_{22})\,\xi_1 + \eta_1 + \ldots .$$

Hence if $\gamma'_{22} \neq 0$, we can make the coefficient of ξ_1 zero. But if $\gamma'_{22} = 0$, then $\beta'_{11} = 0$. Hence if $m > 2$, I contains a substitution, leaving ξ_1 fixed and replacing η_1 by $\eta_1 + \beta''_{12}\eta_2 + \alpha''_{12}\xi_2$. Then, by (12_r), $\alpha''_{12}\beta''_{12} = 0$. Transforming by M_1M_2, if necessary, we can suppose that $\beta''_{12} = 0$, so that we are led to case (b_2).

For $m = 2$, I contains the substitution S_2,

$$\xi'_1 = \xi_1, \quad \eta'_1 = \beta_{11}\xi_1 + \eta_1 + \mu(\xi_2 + \eta_2), \quad \xi'_2 = \alpha_{21}\xi_1 + \alpha_{22}\xi_2 + \gamma_{22}\eta_2, \text{ etc.}$$

We may suppose $\gamma_{22} \neq 0$, since otherwise, $\alpha_{21} = 0$ and then $\alpha_{22} = 0$ by (11_r). Transforming S_2 by $R_{1,2,\kappa}$ we obtain a substitution in I which leaves ξ_1 fixed and replaces η_1 by

$$(\beta_{11} + \kappa\alpha_{21} + \kappa\mu + \kappa^2\gamma_{22})\,\xi_1 + \eta_1 + (\mu + \kappa + \kappa\alpha_{22} + \lambda\mu\kappa^2 + \lambda\kappa^3\gamma_{22})\,\xi_2 + (\mu + \kappa\gamma_{22})\,\eta_2$$

We may therefore make the coefficient of η_2 zero, whence we are led to case (a) or case (b_2).

b_2). If $\beta'_{12} = 0$, then $\beta'_{11} + \lambda\beta'^{\,2}_{11} = 0$. Consider the case $m > 2$. If J has a substitution T, leaving ξ_1, η_1 and ξ_2 fixed, then $S'_1 = S_1^{-1}T^{-1}S_1T$ leaves ξ_1 and η_1 fixed. The proposition therefore follows unless S'_1 is the identity for every possible T. But if S_1 be commutative with $R_{2,3,\kappa}$ and $Q_{3,2,\kappa}$, it must have the form

$$S_1:\ \begin{cases} \xi'_1 = \xi_1, \quad \eta'_1 = \beta'_{11}\xi_1 + \eta_1 + \xi_2, \\ \xi'_2 = \xi_2, \quad \eta'_2 = \xi_1 + \beta'_{22}\xi_2 + \eta_2 + \beta'_{23}\xi_3 + \delta'_{23}\eta_3 + \ldots, \\ \xi'_3 = \delta'_{23}\xi_2 + \xi_3, \quad \eta'_3 = \beta'_{23}\xi_2 + \eta_3. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

If $m > 3$, by supposing S_1 commutative with $R_{3,4,\kappa}$, $Q_{4,3,\kappa}$, etc., we readily see that it reduces to a substitution affecting only $\xi_1, \eta_1, \xi_2, \eta_2$, leading to the case $m = 2$, treated below.

If $m = 3$, $n > 1$, a mark $\kappa \neq 0, \neq 1$ exists in the $GF[2^n]$. If S_1 be commutative with $T_{3,\kappa}$, then $\delta'_{23} = \beta'_{23} = 0$, so that we are led to the case $m = 2$.

If $m = 3$, $n = 1$, we have $\beta_{11} = 0$ by (21). The product $S_2 \equiv M_1M_3S_1M_1M_3S_1$ replaces ξ_3 and η_3 by respectively

$$\xi_3 + (\beta'_{23} + \delta'_{23})\,\xi_2, \quad \eta_3 + (\beta'_{23} + \delta'_{23})\,\xi_2.$$

If $\beta'_{23} = \delta'_{23} = 0$ or 1, we have in S_2 a substitution belonging to I, different from the identity, and leaving ξ_3 and η_3 fixed. If one be zero and the other 1, then S_2 has $\beta''_{23} = \delta''_{23} \equiv \beta'_{23} + \delta'_{23} = 1$. Taking this S_2 in place of our previous S, we evidently obtain the desired result.

For $m = 2$, the substitution S_1, leaving ξ_1 fixed and replacing η_1 by $\beta_{11}\xi_1 + \eta_1 + \xi_2$, where $\beta_{11} = 0$ or λ^{-1}, has for $\beta_{11} = \lambda^{-1}$, the form $S_2 \equiv M_1 M_2 L T_{2,\alpha} Q_{2,1,\alpha}$ and for $\beta_{11} = 0$ the form $S_3 \equiv R_{1,2,1} T_{2\alpha}$. But $T_{2\rho}$ transforms S_2 into $S'_2 \equiv M_1 M_2 L T_{2,\alpha\rho^2} Q_{2,1,\alpha\rho}$. Hence I contains

$$S_2^{-1} S'_2 \equiv Q_{2,1,\alpha} T_{2,\rho^2} Q_{2,1,\alpha\rho} \equiv T_{2,\rho^2} Q_{2,1,\alpha\rho^2+\alpha\rho}.$$

Transforming by $T_{2,\alpha\rho}^{-1}$, we get $T_{2,\rho^2} Q_{2,1,\rho+1}$. This $R_{1,2,\kappa}$ transforms into $R_{1,2,\kappa(\rho^2+1)} T_{2,\rho^2} Q_{2,1,\rho+1}$. Hence I contains $R_{1,2,\kappa(\rho^2+1)} \neq 1$, if $\rho \neq 1, \neq 0$, as we may assume if $n > 1$.

Similarly, the transformed of S_3 by $R_{1,2,1}$ gives $R_{1,2,1} T_{2,\alpha} R_{1,2,\alpha^{-1}}$. Hence I contains $R_{1,2,\alpha^{-1}}$. Then, as in case (a), I contains a $T_{2,\kappa} \neq 1$.

45. Proposition III.—*If* $m > 2$, *the group* I *contains one of the substitutions* $N_{i,j,\kappa}$ $(i, j > 1)$, *not the identity.*

If $m - 1 > 2$, the group $J^{(m-1)}$, composed of all the substitutions of J which leave ξ_1 and η_1 fixed, is a simple[*] group. Therefore the group I, have one such substitution, has all.

For the case $m - 1 = 2$, it follows that I contains $N_{2,3,\kappa}$ or else $P_{23} Q_{3,2,1}$. The existence of a third pair of indices was assumed in §8 of the paper cited only in transforming by a product of two M_i's or in deriving from $P_{12} Q_{2,1,1}$ a substitution $Q_{3,1,1}$ [in case (I_b) of p. 501]. The former operations are allowable in the present investigation since $M_1 M_2$, $M_1 M_3$ belong to our group J.

Transforming $P_{23} Q_{3,2,1}$ by $T_{3,\kappa}$, we get $T_{3,\kappa}^{-1} P_{23} T_{3,\kappa} Q_{3,2,\kappa}$. Hence I contains the product

$$P_{23} T_{2,\kappa}^{-1} T_{3,\kappa} Q_{3,2,\kappa} \cdot Q_{3,2,1} P_{23},$$

and therefore its transformed by P_{23}, giving

$$S_4 \equiv T_{2,\kappa^{-1}}\, T_{3,\kappa} Q_{3,2,\kappa+1}.$$

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[*] Dickson, "The Structure of the Hypoabelian Groups," Bulletin of the American Mathematical Society, July, 1898.

If $\varkappa \neq 1$, as we may suppose if $n > 1$, this substitution is not the identity; similarly for the product

$$S_4^{-1} T_{3\kappa}^{-1} S_4 T_{3\kappa} \equiv Q_{3,2,(\kappa+1)^2}.$$

For the case $n = 1$, we refer to the computation of §18 of the paper cited, where it is proven that I contains $Q_{3,2,1}$.

46. We may now prove directly that the invariant subgroup I contains the generators L, $M_i M_j$, $N_{i,j,\kappa}$ of J, so that J is simple.

For $m > 2$, we employ the substitution* derived from the W of §44,

$$V \equiv T_{3,\lambda^{-\frac{1}{2}}} T_{2,\lambda^{\frac{1}{2}}} W M_1 M_2,$$

$$V: \begin{cases} \xi_1' = \lambda^{-\frac{1}{2}} \eta_2 + \lambda^{\frac{1}{2}} (\xi_3 + \eta_3) \quad , & \eta_1' = \lambda^{\frac{1}{2}} (\xi_2 + \eta_2) \quad , \\ \xi_2' = \lambda \xi_1 + \eta_1 + \lambda^{\frac{3}{2}} (\xi_3 + \eta_3) \quad , & \eta_2' = \xi_1 + \lambda^{\frac{1}{2}} (\xi_3 + \eta_3) \quad , \\ \xi_3' = \eta_1 + \lambda^{\frac{1}{2}} (\xi_2 + \eta_2) + \lambda^{-\frac{1}{2}} \xi_3, & \eta_3' = \lambda \eta_1 + \lambda^{\frac{3}{2}} (\xi_2 + \eta_2) + \lambda^{\frac{1}{2}} \eta_3. \end{cases}$$

We verify that V transforms $M_2 M_3$ into $L M_1 M_3 T_{3,\lambda^{-1}}$, so that I contains $L M_1 M_3$. Further, I contains the product

$$Q_{2,3,\lambda^{-1}} P_{23} M_2 M_3 = \begin{cases} \xi_2' = \lambda^{-1} \eta_2 + \eta_3, & \eta_2' = \xi_3 \quad , \\ \xi_3' = \eta_2 \quad , & \eta_3' = \xi_2 + \lambda^{-1} \xi_3, \end{cases}$$

which is transformed by V into the substitution

$$\begin{cases} \xi_1' = \eta_1, \quad \xi_2' = (\lambda + 1) \xi_2 + \lambda^2 \eta_2 + (\lambda^2 + \lambda) \xi_3 + \lambda \eta_3, \\ \eta_1' = \xi_1, \quad \eta_2' = \xi_2 + (\lambda + 1) \eta_2 + (\lambda + 1) \xi_3 + \eta_3, \\ \xi_3' = \xi_2 + \lambda \eta_2 + \lambda \xi_3 + \eta_3, \quad \eta_3' = (\lambda + 1) \xi_2 + (\lambda^2 + \lambda) \eta_2 + (\lambda^2 + 1) \xi_3 + \lambda \eta_3. \end{cases}$$

This substitution is seen to be the product

$$M_1 M_3 Q_{3,2,1} N_{3,2,\lambda} Q_{2,3,\lambda} R_{2,3,1}.$$

Hence I contains $M_1 M_3$ and therefore also L. But

$$(L M_1 M_3)^{-1} R_{1,2,\kappa} (L M_1 M_3) R_{1,2,\kappa} = Q_{1,2,\lambda^{-1}\kappa}.$$

It follows now that I contains all the generators of J.

For $m = 2$, $n > 1$, we have proven that I contains a $T_{2,\rho} \neq 1$. Transforming it by $N_{1,2,\kappa}$ we obtain (as in §43) the substitution $N_{1,2,\kappa+\kappa\rho^{-1}} T_{2,\rho}$. Hence I

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*V corresponds to the substitution of Jordan, p. 211, l. 18, denoted by French capital U.

contains $N_{1,2,\kappa+\kappa\rho^{-1}}$, not the identity. Transforming by $T_{2,\sigma}$ we reach every $N_{1,2,\kappa}$. Transforming $N_{1,2,\kappa}$ by L and LM_1M_2 we obtain $Q_{2,1,\kappa}$ and $Q_{1,2,\kappa}$ respectively. As in §44, case (a), I contains every $T_{2,\kappa}$. By (13) it contains LM_1M_2. If $n > 1$, we may assume that $\lambda \neq 1$. Setting $\tau = \dfrac{1}{1+\lambda}$, we find

$$Q_{2,1,1}Q_{1,2,1}T_{2\lambda}Q_{2,1,1}Q_{1,2,1} = LR_{1,2,\tau}Q_{2,1,\tau^{-1}}T_{2,\tau^2},$$

Hence I contains L and therefore M_1M_2. Hence $I \equiv J$.

For $m = 2$, $n = 1$, the group* J is the simple icosahedral group of order 60.

Linear homogeneous group Γ in the $GF[2^n]$ in $2m+1$ indices, defined by a quadratic invariant, §§47–48.

47. By §32, we may give the invariant the canonical form

$$\psi \equiv \xi_0^2 + \sum_{i=1}^{m} \xi_i\eta_i .$$

The conditions that a substitution

$$S:\ \begin{cases} \xi_i' = \varkappa_i\xi_0 + \sum\limits_{j=1}^{m} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j), \\ \eta_i' = \sigma_i\xi_0 + \sum\limits_{j=1}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j), \qquad (i = 1, 2, \ldots, m) \\ \xi_0' = \varkappa_0\xi_0 + \sum\limits_{j=1}^{m} (\alpha_{0j}\xi_j + \gamma_{0j}\eta_j), \end{cases}$$

shall leave ψ absolutely invariant are seen to be the relations (11) of §34, together with the following:

$$\sum_{i=1}^{m} (\varkappa_i\beta_{ik} + \sigma_i\alpha_{ik}) = 0, \quad \sum_{i=1}^{m} (\varkappa_i\delta_{ik} + \sigma_i\gamma_{ik}) = 0, \qquad (25)$$

$$(k = 1, 2, \ldots, m)$$

$$\alpha_{0j}^2 = \sum_{i=1}^{m} \alpha_{ij}\beta_{ij}, \quad \gamma_{0j}^2 = \sum_{i=1}^{m} \gamma_{ij}\delta_{ij}, \quad \varkappa_0^2 + \sum_{i=1}^{m} \varkappa_i\sigma_i = 1. \qquad (25')$$

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* Bulletin of the American Math. Soc., pp. 508–9, July, 1898.

It is known* that, for every set of solutions α_{ij}, β_{ij}, γ_{ij}, δ_{ij} in the $GF[2^n]$ of the relations (11), there exists an Abelian substitution

$$\Sigma: \left\{ \begin{aligned} \xi_i' &= \sum_{j=1}^{m} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j), \\ \eta_i' &= \sum_{j=1}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j) \end{aligned} \right. \qquad (i = 1, \ldots, m)$$

of determinant $\Delta \neq 0$ in the field. It is interesting to verify directly that $\Delta \neq 0$. Indeed, suppose that

$$\Delta \equiv \begin{vmatrix} \alpha_{11} & \gamma_{11} & \ldots & \alpha_{1m} & \gamma_{1m} \\ \beta_{11} & \delta_{11} & \ldots & \beta_{1m} & \delta_{1m} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha_{m1} & \gamma_{m1} & \ldots & \alpha_{mm} & \gamma_{mm} \\ \beta_{m1} & \delta_{m1} & \ldots & \beta_{mm} & \delta_{mm} \end{vmatrix} = 0 .$$

We could then suppose that, for example,

$$\begin{aligned} \gamma_{im} &= \sum_{j=1}^{m} \lambda_j \alpha_{ij} + \sum_{j=1}^{m-1} \mu_j \gamma_{ij}, \\ \delta_{im} &= \sum_{j=1}^{m} \lambda_j \beta_{ij} + \sum_{j=1}^{m-1} \mu_j \delta_{ij}. \end{aligned} \qquad (i = 1, \ldots, m)$$

But these values do not satisfy the relation (11),

$$\sum_{i=1}^{m} \begin{vmatrix} \alpha_{im} & \gamma_{im} \\ \beta_{im} & \delta_{im} \end{vmatrix} = 1 .$$

Indeed the left member becomes

$$\sum_{i=1}^{m} \left\{ \sum_{j=1}^{m} \lambda_j \begin{vmatrix} \alpha_{im} & \alpha_{ij} \\ \beta_{im} & \beta_{ij} \end{vmatrix} + \sum_{j=1}^{m-1} \mu_j \begin{vmatrix} \alpha_{im} & \gamma_{ij} \\ \beta_{im} & \delta_{ij} \end{vmatrix} \right\},$$

which, on applying (11), reduces to zero in the $GF[2^n]$.

Since $\Delta \neq 0$, it follows from (25) that

$$x_i = \sigma_i = 0 . \qquad (i = 1, \ldots, m)$$

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* Dickson, "A Triply-infinite System of Simple Groups," The Quarterly Journal, 1897.

Indeed the determinant of the coefficients of the $2m$ linear homogeneous equations (25) is seen to equal Δ. Hence S takes the form

$$S:\quad \begin{cases} \xi_i' = \sum\limits_{j=1}^{m} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j), & \\ \eta_i' = \sum\limits_{j=1}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j), & (i = 1, \ldots, m) \\ \xi_0' = \xi_0 + \sum\limits_{j=1}^{m} \left\{ \left(\sum\limits_{i=1}^{m} \alpha_{ij}\beta_{ij} \right)^{\frac{1}{2}} \xi_j + \left(\sum\limits_{i=1}^{m} \gamma_{ij}\delta_{ij} \right)^{\frac{1}{2}} \eta_j \right\}, & \end{cases}$$

the coefficients being subject to the relations (11) alone. The group of substitutions S is therefore simply isomorphic to the Abelian group of substitutions Σ on $2m$ indices in the $GF[2^n]$. Its structure was determined in the paper cited except when $m = 2$, $n > 1$, in which case the group may be proven to be simple.*

48. Another proof of this result consists in the determination of that subgroup of the first hypoabelian group G_0, leaving $\sum\limits_{i=0}^{m} \xi_i\eta_i$ invariant, for which also the relation $\xi_0 = \eta_0$ is invariant.

In the general substitution of G_0,

$$T:\quad \begin{cases} \xi_i' = \sum\limits_{j=0}^{m} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j), & \\ & (i = 0, 1, \ldots, m) \\ \eta_i' = \sum\limits_{j=0}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j) & \end{cases}$$

we must have

$$\beta_{0j} = \alpha_{0j}, \quad \gamma_{0j} = \delta_{0j}, \quad \alpha_{00} + \gamma_{00} = \beta_{00} + \delta_{00}. \qquad (j = 1, \ldots, m)$$

But the inverse to T is

$$T^{-1}:\quad \begin{cases} \xi_i' = \sum\limits_{j=0}^{m} (\delta_{ji}\xi_j + \gamma_{ji}\eta_j), & \\ & (i = 0, 1, \ldots, m) \\ \eta_i' = \sum\limits_{j=0}^{m} (\beta_{ji}\xi_j + \alpha_{ji}\eta_j). & \end{cases}$$

Putting $\xi_0 = \eta_0$, we find for the coefficients of ξ_0 in ξ_i' and η_i',

$$\delta_{0i} + \gamma_{0i} \equiv 0, \quad \beta_{0i} + \alpha_{0i} \equiv 0. \qquad (i = 1, \ldots, m)$$

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* Quarterly Journal of Mathematics, 1899, vol. XXX, p. 383.

But every substitution S of the group Γ is the inverse T^{-1} of some substitution T belonging to Γ. Hence in S the coefficients of ξ_0 in ξ_i' and η_i' are all zero. By the remaining hypoabelian conditions we see that T must be an Abelian substitution of the form S at the end of §47.

Study of quaternary groups with quadratic invariants. Isomorphisms with known groups; summary; §§49–56.

49. In virtue of the identity

$$\xi_1^2+\xi_2^2+\ldots\ldots+\xi_M^2-\xi_{M+1}^2-\ldots\ldots-\xi_{2M}^2\equiv\sum_{i=1}^{M}(\xi_i-\xi_{M+i})(\xi_i+\xi_{M+i}),$$

it follows from §1 that the group L_{M,p^n}, leaving $\sum\limits_{i=1}^{M} X_i Y_i$ invariant, is simply isomorphic to the group $G_{2M,p^n}^{(M)}$ if -1 be a not-square in the $GF[p^n]$, i. e. if p^n be of the form $4l-1$, but is simply isomorphic to the orthogonal group $G_{2M,p^n}^{(2M)}$ if $p^n=4l+1$.

The structure of the group L_{M,p^n} has been determined directly by the writer, and from the isomorphisms obtained in the paper cited,[*] we derive the following:

Theorem: *The simple groups of order*

$$\tfrac{1}{8}\,\Omega_{6,p^n}^{(6)}\equiv\tfrac{1}{4}(p^{5n}-p^{2n})(p^{4n}-1)\,p^{3n}\,(p^{2n}-1)\,p^n$$
$$\equiv\frac{(p^{4n}-1)(p^{4n}-p^n)(p^{4n}-p^{2n})(p^{4n}-p^{3n})}{4\,(p^n-1)},$$

the one derived from the 6*-ary orthogonal group and the other from the general quaternary linear homogeneous group, each in the* $GF[p^n=4l+1]$, *are simply isomorphic. A like result holds for the simple groups of order*

$$\tfrac{1}{4}\,\Omega_{6,p^n}^{(5)}\equiv\tfrac{1}{2}(p^{5n}-p^{2n})(p^{4n}-1)\,p^{3n}\,(p^{2n}-1)\,p^n,$$

the one derived from the group $G_{6,p^n}^{(5)}$ *and the other from the general quaternary linear homogeneous group, each in the* $GF[p^n=4l-1]$. *Likewise, the simple group* J_0, *a subgroup of index two under the first hypoabelian group on* $m=3$ *pairs of indices,*

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[*] "The Structure of Certain Linear Groups with Quadratic Invariants," Proceedings of the London Mathematical Society, vol. XXX, pp. 70–98, 1899.

and the simple group of quaternary linear homogeneous substitutions of determinant unity in the $GF[2^n]$, are isomorphic and of orders

$$(2^{3n}-1)[(2^{4n}-1)\,2^{4n}][(2^{2n}-1)\,2^{2n}] \equiv \frac{(2^{4n}-2^{0})(2^{4n}-2^n)(2^{4n}-2^{2n})(2^{4n}-2^{3n})}{2^n-1}\,.$$

50. We next determine the structure of the group $L_{2,\,p^n}$, leaving absolutely invariant $\xi_1\eta_1+\xi_2\eta_2$. The two sets of generators on the ruled surface

$$\xi_1\eta_1+\xi_2\eta_2=0$$

are given by the two pairs of equations

$$\xi_1+x\xi_2=0,\quad \eta_2-x\eta_1=0, \tag{26}$$

$$\xi_1+x\eta_2=0,\quad \xi_2-x\eta_1=0. \tag{26'}$$

The most general quaternary linear homogeneous substitution, leaving invariant the pair of equations (26), for every value of x in the field, is readily seen to be

$$\begin{cases}\xi_1'=\alpha\xi_1+\gamma\eta_2, & \xi_2'=-\gamma\eta_1+\alpha\xi_2,\\ \eta_1'=\delta\eta_1-\beta\xi_2, & \eta_2'=\beta\xi_1+\delta\eta_2,\end{cases} \tag{27}$$

having the determinant $(\alpha\delta-\beta\gamma)^2$. For it we have

$$\xi_1'+x\xi_2'=\alpha\,(\xi_1+x\xi_2)+\gamma\,(\eta_2-x\eta_1),$$

$$\eta_2'-x\eta_1'=\beta\,(\xi_1+x\xi_2)+\delta\,(\eta_2-x\eta_1).$$

The group of the substitutions (27) is therefore simply isomorphic to the binary group on the variables $\xi_1+x\xi_2$ and $\eta_2-x\eta_1$. Since the transposition $M_2\equiv(\xi_2\eta_2)$ transforms the pair of equations (26) into the pair (26′), we obtain the most general linear homogeneous substitution, leaving invariant the pair of equations (26′), for every x, if we transform the substitution (27) by M_2, giving

$$\begin{cases}\xi_1'=\alpha\xi_1+\gamma\xi_2, & \xi_2'=\beta\xi_1+\delta\xi_2,\\ \eta_1'=\delta\eta_1-\beta\eta_2, & \eta_2'=-\gamma\eta_1+\alpha\eta_2.\end{cases} \tag{28}$$

The product of an arbitrary substitution (27) and an arbitrary substitution (28) gives

$$\begin{pmatrix} \alpha & 0 & 0 & \gamma \\ 0 & \delta & -\beta & 0 \\ 0 & -\gamma & \alpha & 0 \\ \beta & 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} A & 0 & C & 0 \\ 0 & D & 0 & -B \\ B & 0 & D & 0 \\ 0 & -C & 0 & A \end{pmatrix} = \begin{pmatrix} \alpha A & -\gamma C & \alpha C & \gamma A \\ -\beta B & \delta D & -\beta D & -\delta B \\ \alpha B & -\gamma D & \alpha D & \gamma B \\ \beta A & -\delta C & \beta C & \delta A \end{pmatrix}. \qquad (29)$$

The same result holds if the substitutions be compounded in reverse order, so that the substitutions are commutative. Further, the only substitutions belonging to both of the sets (27) and (28) are seen to be

$$\xi_1' = \alpha\xi_1, \quad \eta_1' = \alpha\eta_1, \quad \xi_2' = \alpha\xi_2, \quad \eta_2' = \alpha\eta_2. \qquad (30)$$

The substitution (27) leaves $\xi_1\eta_1 + \xi_2\eta_2$ absolutely invariant if and only if $\alpha\delta - \beta\gamma = 1$. Hence there are $(p^{2n} - 1)p^n$ such substitutions. It follows that there are

$$\{(p^{2n} - 1)p^n\}^2, \qquad (\text{if } p = 2)$$
$$\tfrac{1}{2}\{(p^{2n} - 1)p^n\}^2 \qquad (\text{if } p > 2)$$

distinct substitutions (29) for which

$$\alpha\delta - \beta\gamma = 1, \quad AD - BC = 1. \qquad (31)$$

The substitution $T_{2,\kappa}$ will be of the form (29) only if

$$\alpha A = \delta D = 1; \quad \alpha D = \kappa, \quad \delta A = \kappa^{-1}, \quad \beta = \gamma = B = C = 0.$$

Therefore $A = \alpha^{-1}$, $D = \kappa\alpha^{-1}$, $\delta = \kappa^{-1}\alpha$, so that

$$\alpha\delta - \beta\gamma = \kappa^{-1}\alpha^2 \quad AD - BC = \kappa\alpha^{-2}.$$

It will thus satisfy the relations (31) only when κ is a square in the $GF[p^n]$. Hence there are at least $\{(p^{2n} - 1)p^n\}^2$ substitutions (29) which satisfy the single relation

$$(\alpha\delta - \beta\gamma)(AD - BC) = 1. \qquad (32)$$

Among these does not occur the transposition $M_1 \equiv (\xi_1 \eta_1)$; for among the conditions that (29) shall reduce to M_1 are found

$$\alpha A = \delta D = 0, \quad \alpha D = \delta A = 1.$$

Since the group $L_{2,\,p^n}$, leaving $\xi_1\eta_1 + \xi_2\eta_2$, is of order $2\{(p^{2n}-1)p^n\}^2$, the group $L'_{2,\,p^n}$ of the substitutions (29) which satisfy (32) is of index two under $L_{2,\,p^n}$. Further, the group $L''_{2,\,p^n}$ of the substitutions (29) which satisfy (31) is of index 2 or 1 under $L'_{2,\,p^n}$ according as $p > 2$ or $p = 2$. But $L''_{2,\,p^n}$ has an invariant subgroup formed of the substitutions (27) which satisfy the relation $\alpha\delta - \beta\gamma = 1$. This subgroup, being simply isomorphic to the group of binary linear substitutions of determinant unity, is for $p = 2$, the group $F_{1,\,p^n}$ of linear fractional substitutions of determinant unity on one index, but for $p > 2$ has the factor groups $F_{1,\,p^n}$ and C, the latter being the group generated by the substitution changing the sign of every index. The quotient group of $L''_{2,\,p^n}$ by the group of substitutions (27) is evidently $F_{1,\,p^n}$. Now $F_{1,\,p^n}$ is *simple* if p^n is neither 2 nor 3.

Theorem:* *The factors of composition of $L_{2,\,p^n}$ are*

(if $p > 2$) $\quad 2,\ 2,\ \frac{1}{2}(p^{2n}-1)p^n,\ \ \frac{1}{2}(p^{2n}-1)p^n,\ \ 2,$
(if $p = 2$) $\quad 2,\ (2^{2n}-1)2^n,\ \ (2^{2n}-1)2^n,$

except when $p_n = 2$ or 3, when the composite numbers 6 *and* 12 *respectively are to be replaced by their prime factors.*

51. Theorem: *For $p^n > 3$, the group $G^{(8)}_{4,\,p^n}$, leaving invariant*

$$\phi \equiv \zeta_1^2 + \zeta_2^2 + \zeta_3^2 + \nu\zeta_4^2, \qquad (\nu = \text{not-square})$$

is simply isomorphic to the group $E_{4,\,p^n}$, leaving invariant

$$f \equiv \xi_1\eta_1 + \xi_2\eta_2 + \lambda(\xi_1^2 + \eta_1^2),$$

where $\xi_1\eta_1 + \lambda\xi_1^2 + \lambda\eta_1^2$ is irreducible in the $GF[p^n]$.

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* We readily verify the statement in §40 that, for $m = 2$, J_0 requires other generators than M_1M_2, $N_{1,\,2,\,\kappa}$. Indeed, every product derived from these two substitutions is of the form SM_1M_2, where S is derived from $N_{1,\,2,\,\kappa}$ and $R_{1,\,2,\,\kappa}$, each of which is of the form (27). Hence the group G generated by M_1M_2 and $N_{1,\,2,\,\kappa}$ is a subgroup of the group of substitutions (27) when extended by M_1M_2. Its order is therefore a factor of $2(2^{2n}-1)2^n$, and hence $< \{(2^{2n}-1)2^n\}^2$.

Suppose first that -1 is the square of a mark I belonging to the field. Then the substitution

$$\xi_2 = \zeta_1 + I\zeta_2, \quad \eta_2 = \zeta_1 - I\zeta_2,$$

transforms ϕ into

$$\phi_1 \equiv \xi_2\eta_2 + \zeta_3^2 + \nu\zeta_4^2.$$

Applying to ϕ_1 the substitution of determinant $2\alpha\beta$,

$$\zeta_3 = \alpha(\xi_1 - \eta_1), \quad \zeta_4 = \beta(\xi_1 + \eta_1), \tag{33}$$

we obtain the function

$$\xi_2\eta_2 + (2\nu\beta^2 - 2\alpha^2)\,\xi_1\eta_1 + (\alpha^2 + \nu\beta^2)(\xi_1^2 + \eta_1^2),$$

which may be made to assume the form f. Indeed, by §3, there exist $p^n + 1$ sets of solutions in the $GF[p^n]$ of

$$2\nu\beta^2 - 2\alpha^2 = 1.$$

At most, two of these sets of solutions make $\alpha\beta = 0$; for, $\alpha = 0$ gives a solution only when 2 is a not-square, in which case $\beta = 0$ is not a solution. Hence there are $p^n - 1$ substitutions (33) of determinant not zero which transform ϕ_1 into f

Suppose, however, that -1 is not a not-square in the field. We may take $\nu = -1$. Applying to ϕ the substitution of determinant $\alpha\beta$,

$$\zeta_1 = \alpha(\xi_1 - \eta_1), \quad \zeta_2 = \beta(\xi_1 + \eta_1), \quad \zeta_3 = \tfrac{1}{2}(\eta_2 + \xi_2), \quad \zeta_4 = \tfrac{1}{2}(\eta_2 - \xi_2),$$

we obtain the function

$$\xi_2\eta_2 + (2\beta^2 - 2\alpha^2)\,\xi_1\eta_1 + (\alpha^2 + \beta^2)(\xi_1^2 + \eta_1^2).$$

But there exist, in the $GF[p^n]$, $p^n - 1$ sets of solutions of

$$2\beta^2 - 2\alpha^2 = 1.$$

Two of these sets make $\alpha\beta = 0$. Hence there are $p^n - 3$ substitutions of determinant not zero which reduce ϕ to the form f.

For $p^n = 3$, there are no quadratic forms

$$q \equiv \xi_1\eta_1 + \lambda\xi_1^2 + \lambda\eta_1^2,$$

irreducible in the $GF[p^n]$. Indeed, according as $\lambda = +1$ or -1, q becomes $(\xi_1 - \eta_1)^2$ or $-(\xi_1 + \eta_1)^2$.

52. Denote by E'_{4,p^n} the subgroup which $M_1 \equiv (\xi_1\eta_1)$ extends to the total group E_{4,p^n}. The order of E'_{4,p^n} is

$$(p^{3n}+p^n)(p^{2n}-1)\,p^n \equiv (p^{4n}-1)\,p^{2n}.$$

If $p=2$, the group E'_{4,p^n} is identical with the second hypoabelian group $G_{\lambda'}$ on two pairs of indices. It will be evident from what follows that the group E'_{4,p^n}, for $p>2$, has a subgroup E''_{4,p^n} of index two which is extended to E' by the substitution $T_{2,N}$, where N is a not-square in the $GF[p^n]$. We may verify this result directly. Thus, if -1 be a not-square, the substitution $T_{2,-1}$ of E' corresponds to the substitution

$$\zeta_3' = -\zeta_3, \quad \zeta_4' = -\zeta_4 \tag{34}$$

of the group $G^{(3)}_{4,p^n}$, leaving ϕ invariant. If -1 be the square of a mark I in the field, the substitution $T_{2,N}$ corresponds to the substitution of $G^{(3)}_{4,p^n}$,

$$\begin{cases} \zeta_1' = \tfrac{1}{2}\,(N+N^{-1})\,\zeta_1 + \tfrac{1}{2}\,I(N-N^{-1})\,\zeta_2, \\ \zeta_2' = -\tfrac{1}{2}\,I(N-N^{-1})\,\zeta_1 + \tfrac{1}{2}\,(N+N^{-1})\,\zeta_2, \end{cases} \tag{35}$$

which is an orthogonal substitution, leaving $\zeta_1^2+\zeta_2^2$ invariant, but not of the form $Q^{\alpha,\beta}_{1,2}$ since

$$2\alpha^2 - 1 = \tfrac{1}{2}\,(N+N^{-1})$$

would require $\alpha^2 = (N+1)^2/4N$, a not-square.

By §§15–17 the substitution (34) or (35) respectively serves to extend a subgroup H to $G^{(3)}_{4,p^n}$.

For $p=2$, we set $E'' \equiv E'$.

53. Theorem: *The group E''_{4,p^n} is simply isomorphic to the group of linear fractional substitutions of determinant unity.*

We transform the invariant f into $XY+\xi_2\eta_2$ by means of the following substitution of determinant $2\sigma+1$,

$$Z:\ \begin{cases} X = \lambda\xi_1 - \sigma\eta_1, \\ Y = \xi_1 - \lambda\sigma^{-1}\eta_1, \end{cases}$$

where σ is a root of the equation

$$\sigma^2 + \sigma + \lambda^2 = 0,$$

irreducible in the $GF[p^n]$ in virtue of the irreducibility of

$$(\lambda\xi_1)^2 + (\lambda\xi_1)\eta_1 + \lambda^2\eta_1^2.$$

For the reciprocal of Z we find

$$Z^{-1}:\ \begin{cases}(2\sigma+1)\xi_1 = -\lambda\sigma^{-1}X + \sigma Y,\\ (2\sigma+1)\eta_1 = -\phantom{\lambda\sigma^{-1}}X + \lambda Y.\end{cases}$$

Every substitution S in the $GF[p^n]$, leaving f invariant, is transformed by Z into a substitution S', leaving $XY + \xi_2\eta_2$ invariant, but having its coefficients in the $GF[p^{2n}]$. In particular, Z transforms M_2 and $T_{2,N}$ into themselves. Hence Z transforms the group E''_{4,p^n}, which M_2 and $T_{2,N}$ extend to the total group, leaving f invariant, into a group K which is extended by M_2 and $T_{2,N}$ to the total group, leaving $XY + \xi_2\eta_2$ invariant. It follows from §50 that the substitutions of K are of the form (29), when operating on the indices X, Y, ξ_2, η_2, in which α, β, γ, δ, A, B, C, D are marks of the $GF[p^{2n}]$ satisfying the relations

$$\alpha\delta - \beta\gamma = 1, \quad AD - BC = 1. \tag{36}$$

Expressing the substitution (29) in terms of the indices ξ_1, η_1, ξ_2, η_2, it is seen to take the form:

$$\left\{\begin{array}{l}\xi_1' = \sum\limits_j^{1,2}(\alpha_{1j}\xi_j + \gamma_{1j}\eta_j), \quad \eta_1' = \sum\limits_j^{1,2}(\beta_{1j}\xi_j + \delta_{1j}\eta_j),\\ \xi_2' = (\lambda\alpha B - \gamma D)\xi_1 - (\sigma\alpha B - \lambda\sigma^{-1}\gamma D)\eta_1 + \alpha D\xi_2 + \gamma B\eta_2,\\ \eta_2' = (\lambda\beta A - \delta C)\xi_1 - (\sigma\beta A - \lambda\sigma^{-1}\delta C)\eta_1 + \beta C\xi_2 + \delta A\eta_2,\end{array}\right\} \tag{37}$$

where we have written for brevity

$$\begin{aligned}
\alpha_{11} &= (2\sigma+1)^{-1}(\sigma\delta D - \lambda\sigma\beta B + \lambda\sigma^{-1}\gamma C - \lambda^2\sigma^{-1}\alpha A),\\
\gamma_{11} &= (2\sigma+1)^{-1}(\lambda\alpha A - \lambda\delta D + \sigma^2\beta B - \lambda^2\sigma^{-2}\gamma C),\\
\beta_{11} &= (2\sigma+1)^{-1}(\gamma C + \lambda\delta D - \lambda\alpha A - \lambda^2\beta B),\\
\delta_{11} &= (2\sigma+1)^{-1}(\sigma\alpha A + \lambda\sigma\beta B - \lambda\sigma^{-1}\gamma C - \lambda^2\sigma^{-1}\delta D),\\
\alpha_{12} &= (2\sigma+1)^{-1}(-\sigma\beta D - \lambda\sigma^{-1}\alpha C), \quad \gamma_{12} = (2\sigma+1)^{-1}(-\sigma\delta B - \lambda\sigma^{-1}\gamma A),\\
\beta_{12} &= (2\sigma+1)^{-1}(-\alpha C - \lambda\beta D) \quad\ , \quad \delta_{12} = (2\sigma+1)^{-1}(-\gamma A - \lambda\delta B).
\end{aligned}$$

We next require that all of the coefficients of the substitution (37) shall belong to the $GF[p^n]$. The totality of substitutions thus obtained form the group K simply isomorphic to E''_{4,p^n}.

54. Since σ belongs to the $GF[p^{2n}]$, but not to the $GF[p^n]$, we may set

$$\alpha = a + a'\sigma, \quad \beta = b + b'\sigma, \quad \gamma = c + c'\sigma, \quad \delta = d + d'\sigma.$$

The coefficient δA must belong to the $GF[p^n]$. If $d' \neq 0$, we may set $A = \varkappa + A_1 d'\sigma$, where $\varkappa$ and A_1 are marks of the $GF[p^n]$. Applying $\sigma^2 + \sigma + \lambda^2 = 0$, we find

$$\delta A = (\varkappa d - \lambda^2 A_1 d'^2) + \sigma(\varkappa + dA_1 - A_1 d')\, d'.$$

Hence must $\varkappa = A_1 d' - dA_1$. If $d' = 0$, $d \neq 0$, we may evidently set $A = -dA_1$, a mark of the field. Finally, if $d = d' = 0$, so that $\delta = 0$, the coefficients of ξ_1 and η_1 in η_2' require that $\lambda\beta A$ and $-\sigma\beta A$ be marks of the $GF[p^n]$ and hence require that $\beta A = 0$. Since $\alpha\delta - \beta\gamma \neq 0$, we must have $\beta\gamma \neq 0$ and therefore $A = 0$. Hence in every case we may set $A = (d' - d + d'\sigma) A_1$.

Also γB, βC, δA must belong to the $GF[p^n]$. Proceeding as before, we and that we may set

$$A = (d' - d + d'\sigma) A_1, \quad B = (c' - c + c'\sigma) B_1,$$
$$C = (b' - b + b'\sigma) C_1, \quad D = (a' - a + a'\sigma) D_1,$$

where A_1, B_1, C_1 and D_1 belong to the $GF[p^n]$.

We next set up the conditions that the remaining coefficients of the substitution (37) shall belong to the $GF[p^n]$. Expressing the coefficients $\lambda\beta A - \delta C$ and $-\sigma\beta A + \lambda\sigma^{-1}\delta C$ in the form $R + S\sigma$ and setting the coefficient of σ equal zero, we obtain respectively

$$(b'd - bd')(\lambda A_1 + C_1) = 0, \quad (bd - b'd + \lambda^2 b'd')(\lambda A_1 + C_1) = 0.$$

Hence either $\lambda A_1 + C_1 = 0$ or else $\delta C = \beta A = 0$. Consider the latter alternative. If $\delta \neq 0$, then $C = 0$, and therefore $A \neq 0$, since $AD - BC \neq 0$. Hence $\beta = 0$, i. e. $b = b' = 0$. We may therefore give to C_1 an arbitrary value in the $GF[p^n]$ and in particular a value making $\lambda A_1 + C_1 = 0$. If, however, $\delta = 0$, an arbitrary value in the field may be assigned to A_1, so that again we may take $\lambda A_1 + C_1 = 0$. Hence, in every case, $\lambda A_1 + C_1 = 0$.

By a simple interchange of letters, it follows that the coefficients $\lambda\alpha B - \gamma D$ and $-\sigma\alpha B + \lambda\sigma^{-1}\gamma D$ will belong to the $GF[p^n]$ if and only if $\lambda B_1 + D_1 = 0$.

In order that $(2\sigma+1)^{-1}(R+\sigma S)$ shall belong to the $GF[p^n]$, when R and S do, it is necessary and sufficient that $S=2R$. Hence the coefficients denoted by δ_{13} and γ_{13} will belong to the field if and only if respectively

$$\begin{aligned}&(A_1+\lambda B_1)(2cd+2c'd'\lambda^2-c'd-cd')=0,\\&(A_1+\lambda B_1)[cd-cd'+c'd'\lambda^2-2\lambda^2(c'd-cd')]=0.\end{aligned}$$

If $A_1+\lambda B_1\neq 0$, we find the relation $(c'd-cd')(1-4\lambda^2)=0$. But, for $p>2$, $\lambda\neq\frac{1}{2}$, since then $\sigma^2+\sigma+\lambda^2=(\sigma+\frac{1}{2})^2$. Hence

$$c'd-cd'=0,\quad cd-cd'+c'd'\lambda^2=0,$$

so that $\gamma A=\delta B=0$. By the reasoning given above, we may assume that, in every case, $A_1+\lambda B_1=0$.

If we consider the coefficients α_{12} and β_{12}, a simple interchange of letters gives the result $C_1+\lambda D_1=0$ as the condition that α_{12} and β_{12} belong to the $GF[p^n]$.

We have now obtained the following results:

$$C_1=-\lambda A_1,\quad B_1=-\lambda^{-1}A_1,\quad D_1=A_1. \tag{38}$$

In virtue of these relations we may verify that the coefficients α_{11}, γ_{11}, β_{11}, δ_{11} belong to the $GF[p^n]$. The conditions for β_{11} and δ_{11} are respectively

$$\begin{aligned}&(2bc+2\lambda^2b'c'-b'c-bc')(C_1-\lambda^2B_1)\\&\qquad\qquad+(2ad+2\lambda^2a'd'-a'd-ad')(\lambda D_1-\lambda A_1)=0,\\&[a'd-ad-\lambda^2a'd'+2\lambda^2(ad'-a'd)][A_1-D_1]\\&\qquad\qquad+[cb'-cb-\lambda^2c'b'+2\lambda^2(c'b-cb')][\lambda B_1-\lambda^{-1}C_1]=0.\end{aligned}$$

As to the coefficients α_{11} and γ_{11}, we observe that

$$\begin{aligned}\gamma_{11}+\beta_{11}&=(2\sigma+1)^{-1}(1-\lambda^2\sigma^{-2})(\gamma C+\sigma^2\beta B)=\sigma^{-1}(\gamma C+\sigma^2\beta B);\\ \alpha_{11}+\delta_{11}&=(2\sigma+1)^{-1}(\sigma-\lambda^2\sigma^{-1})(\alpha A+\delta D)\quad=\alpha A+\delta D.\end{aligned}$$

These sums will belong to the $GF[p^n]$ if respectively

$$(b'c-bc-\lambda^2b'c')(B_1-\lambda^{-2}C_1)=0,\quad(a'd-ad')(D_1-A_1)=0.$$

55. The condition $\alpha\delta-\beta\gamma=1$ requires that

$$\begin{cases}ad-bc-\lambda^2a'd'+\lambda^2b'c'=1,\\ad'+a'd-a'd'=bc'+b'c-b'c'.\end{cases} \tag{39}$$

In virtue of these relations we find that

$$AD - BC = (\sigma + 1)(a'd' - ad' - a'd)(A_1D_1 - B_1C_1) \\ + (ad - \lambda^2 a'd')(A_1D_1 - B_1C_1) + B_1C_1.$$

Applying (38), we find

$$AD - BC = B_1C_1 = A_1^2.$$

Hence, from (36), $A_1 = \pm 1$.

But the substitution (29) is unaltered by a simultaneous change of sign in a, a', b, b', c, c', d, d'. Hence we may set

$$A_1 = +1, \quad C_1 = -\lambda, \quad B_1 = -\lambda^{-1}, \quad D_1 = 1.$$

It follows that every substitution (29) of the group K is the product UV of two substitutions

$$U \equiv \begin{bmatrix} a + a'\sigma & 0 & 0 & c + c'\sigma \\ 0 & d + d'\sigma & -(b + b'\sigma) & 0 \\ 0 & -(c + c'\sigma) & a + a'\sigma & 0 \\ b + b'\sigma & 0 & 0 & d + d'\sigma \end{bmatrix},$$

$$V \equiv \begin{bmatrix} d' - d + d'\sigma & 0 & -\lambda(b' - b + b'\sigma) & 0 \\ 0 & a' - a + a'\sigma & 0 & \lambda^{-1}(c' - c + c'\sigma) \\ -\lambda^{-1}(c' - c + c'\sigma & 0 & a' - a + a'\sigma & 0 \\ 0 & \lambda(b' - b + b'\sigma) & 0 & d' - d + d'\sigma \end{bmatrix},$$

the coefficients of which must satisfy the relations (39). Now U and V are commutative and are identical only when each is the identity. Hence the group of the products UV is isomorphic to the group of the substitutions U. For $p > 2$ the isomorphism is (1, 2); indeed, a change of sign of a, a', etc., alters U but not the product UV; while, further, UV is the identity only when

$$B = C = \beta = \gamma = 0, \quad A = D, \quad \alpha = \delta, \quad \alpha A = 1,$$

whence $\alpha = \delta = A = D = \pm 1$, giving two [distinct if $p > 2$] substitutions U. By §50 the group of the substitutions U has (1, 2) isomorphism if $p > 2$, but simple isomorphism if $p = 2$, with the group F_{1,p^n} of linear fractional substitutions of determinant unity. Hence the group K of the substitutions UV, and therefore the group E''_{4,p^n}, is simply isomorphic to the simple group F_{1,p^n}.

56. We conclude with a summary of the simple groups obtained—

$$(2^{nm}-1)[(2^{2n(m-1)}-1)\,2^{2n(m-1)}]\,\ldots\,[(2^{2n}-1)\,2^{2n}]. \qquad (m>2)$$

$$(2^{nm}+1)[(2^{2n(m-1)}-1)\,2^{2n(m-1)}]\,\ldots\,[(2^{2n}-1)\,2^{2n}]. \qquad (m>1)$$

$$\tfrac{1}{2}\,(p^{n(m-1)}-1)\,p^{n(m-2)}(p^{n(m-3)}-1)\,p^{n(m-4)}\,\ldots\,(p^{2n}-1)\,p^{n},$$
$$(p>2,\ m \text{ odd and } >1;\ \text{exception } p^n=3,\ m=3).$$

$$\tfrac{1}{4}\,[p^{n(m-1)}-\varepsilon^{\frac{m}{2}}p^{n\left(\frac{m}{2}-1\right)}](p^{n(m-2)}-1)\,p^{n(m-3)}\,\ldots\,(p^{2n}-1)\,p^{n},$$
$$(p>2,\ m \text{ even and } >4).$$

$$\tfrac{1}{2}\,[p^{n(m-1)}+\varepsilon^{\frac{m}{2}}p^{n\left(\frac{m}{2}-1\right)}](p^{n(m-2)}-1)\,p^{n(m-3)}\,\ldots\,(p^{2n}-1)\,p^{n},$$
$$(p>2,\ m \text{ even and } >2).$$

Here $\varepsilon=\pm 1$ according as p^n is of the form $4l\pm 1$. The first and second sets are obtained from the first and second hypoabelian groups; the third and fourth sets from the orthogonal group, and the fifth set from the group $G^{(m-1)}_{m,\,p^n}$.

UNIVERSITY OF CALIFORNIA, *December* 30, 1898.

Upon the Ruled Surfaces Generated by the Plane Movements whose Centrodes are Congruent Conics Tangent at Homologous Points.[*]

By Dr. E. M. Blake, *Brooklyn, N. Y.*

The movements considered in this paper are defined as follows: Upon a plane α' containing a conic C' moves a coincident plane α, containing a conic C congruent to C', in such a manner that C and C' are always tangent at homologous points, i. e. C and C' are the centrodes of the movement. The locus of a point rigidly attached to α is a curve of the fourth order when C and C' are central conics and of the third order when they are parabolas. The locus is in a plane parallel to α' and the same distance from it that the generating point is from α. The locus of a straight line carried by α and making an angle with it, is a quartic scroll when the centrodes are central conics and a cubic scroll when they are parabolas.

It is the object of the present paper to describe the forms of these scrolls, and the character and situation of their nodal lines and pinch-points. The results are to be regarded from two points; first, as furnishing a method of mechanically generating certain cubic and quartic scrolls; and second, as exhibiting the totality of line loci of the movements considered. In the respects mentioned the results are believed to be new.

The properties of the surfaces are deduced from those of their sections parallel to α', i. e. by viewing the surfaces as built up of the loci of the individual points of the generatrix. The point loci have from time to time been studied, and it has only been found necessary in what follows to state briefly the results obtained.

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[*] Read at the meeting of the American Mathematical Society, Boston, Aug. 19–20, 1898. Thread models of the thirteen types of these surfaces were exhibited.

The paper contains three sections devoted respectively to the movements whose centrodes are ellipses, hyperbolas and parabolas. The scrolls described in the first two are of the fourth order, and have a nodal circle of infinite radius in the infinitely distant plane parallel to α' and a nodal straight line intersecting it. The scrolls of the third section are cubic with a nodal straight line.

§1.—*The Centrodes are Ellipses.*

It is easily shown* that, if an ellipse roll upon a congruent one, their points of contact being homologous† points of the two ellipses, each focus of the one remains at a constant distance, viz. the transverse axis of one of the ellipses, from a focus of the other. Denoting the foci of C', the ellipse in α', by F_1' and F_2', the corresponding foci of C by F_1 and F_2, the transverse axis of each of the ellipses by $2a$, and the distance between their foci by $2c$; the theorem shows that the movement produced by the rolling ellipses can also be produced by joining the pairs of points F_1, F_1' and F_2, F_2' by links of length $2a$.

This latter method of defining the movement makes it a special case of "three-bar motion." The investigations of Roberts and Cayley‡ have shown that for the general three-bar motion the locus of a point in the plane α is a sextic curve. Under two conditions these degenerate into a quartic and a conic. One condition is here satisfied; the opposite links are equal in length. The conic generated is a circle of radius $2a$, in describing which any straight line of α moves parallel to itself.

We proceed to give the equation and necessary properties of the locus of any point P of α: Let the transverse and conjugate axes of C' be taken for the axes of x and y respectively and a perpendicular to α' through its centre O' for axis of z of a system of coordinates to which any fixed point—α' is regarded as fixed—can be referred. Let the transverse and conjugate axes of C whose center is O, be taken respectively for the axes of ξ and η of a system of coordinates to which points of α can be referred. Further, let F_1' and F_1—foci which remain at the distance $2a$—be respectively upon the positive halves of the axes of x and ξ. The locus of P whose coordinates in α are (ξ, η) is

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* See Burmester, Kinematik, vol. I, Leipzig, 1888, pp. 302–304, and Clifford, Dynamics, Part I, London, 1898, pp. 146–148.

† Homologous in the sense that by turning one of the ellipses over about a common tangent, the two will be brought into coincidence.

‡ Proc. Lond. Math. Soc., vols. II, III, IV, VII.

$$\begin{aligned}(x^2+y^2)^2 - 2(\xi^2+\eta^2)(x^2+y^2) - 4a^2x^2 + 4(c^2-a^2)y^2 - 8a^2\xi x \\ - 8\eta(c^2-a^2)y + (\xi^2+\eta^2)^2 - 4a^2\xi^2 + 4(c^2-a^2)\eta^2 = 0, \quad (1)\end{aligned}$$

a unicursal curve of the fourth order having (when c is not zero) nodes at the circular points at infinity and a real double point at $(-\xi, \eta)$.* The latter is a node, cusp or conjugate point according as P is without, on or within the ellipse C. The nodes fall on α' outside the ellipse C', the cusps upon it, and the conjugate points within.

It is evident from the mode of generation by the rolling ellipses that a point at a considerable distance from O generates a curve consisting of a loop with a smaller one within joined at a node, similar to the limaçon with a node. As the tracing point still farther recedes from O, the loops become more and more nearly equal passing toward the circular form. The limit for a point at infinity is a double circle. As the tracing point moves from a great distance toward O, both loops of the locus decrease in size, the inner more rapidly than the outer, until, when the ellipse C is reached, the former shrinks to a point. The curve is then similar to the cardioid having a cusp projecting inward. The cusp, as we have seen, is upon C'. Its tangent is normal to C' at the point where the cusp falls. The curve, traced by a point within C, is closed, encircling C' once and having a conjugate point within it. The locus of F_1 is the circle of radius $2a$ whose center is F_1' with the isolated point F_2'; similarly for the locus of F_2.

Under the supposition, as above, that the centrodes are not circles, the only curves having orthogonal symmetry are the loci of the points upon the transverse and conjugate axes of C. They are symmetrical with respect to the corresponding axes of C'. Since they have nodes—not cusps—at the circular points at infinity, they are not limaçon although resembling them. The center O describes the curve

$$(x^2+y^2+2c^2-4a^2)(x^2+y^2) - 2c^2(x^2-y^2) = 0, \qquad (2)$$

symmetrical with respect to both axes.

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*Roberts, Proc. Lond. Math. Soc., III, 1871, p. 220. Cayley, Ibid., IV, p. 110. Dingeldey, "Ueber die Erzeugung von Curven vierter Ordnung durch Bewegungsmechanismen," Leipzig, 1885, p. 11.

The movement with circular centrodes requires brief consideration. By reducing the focal distances of the elliptical centrodes, they pass finally as a limit into circular ones, the equation (1) applying for c equal to zero. The movement can no longer be regarded as a special case of three-bar motion as two of the links reduce to zero.

With elliptical centrodes there is a twofold infinity of non-congruent curves described by the points of α, but with circular ones the distance of a point from the center of the moving centrode alone defines the form of its locus, all points of a circumference concentric with the centrode tracing congruent curves. These curves, as the equation shows, are limaçon,* and hence have cusps at the circular points at infinity.†

The way is now prepared for readily determining the character of the locus of a line carried by α. The generatrix is assumed to be neither parallel nor perpendicular to it. If parallel, the line would envelope a plane curve the study of which is without the field of this paper. If perpendicular, a right cylinder upon the locus of its piercing point in α is the surface generated. Two parallel generatrices having the same orthogonal projection upon α, evidently generate congruent surfaces which can be brought into coincidence by translation along the axis of z. Further, it is to be remarked that the loci of any two intersecting lines having the same projection upon α can be made congruent by multiplying by a constant the ordinates—that are perpendicular to α'—of one of them. Hence, the position in α of the projection of the generatrix of a surface determines its projective properties.

Let the generatrix l_1 pass through A, the foot of the perpendicular from O to l, its orthogonal projection upon α, and let l_1 make the angle with α whose cotangent is s. Take a point P of l whose coordinates are (ξ, η) and let its distance from A be d. Denote the length OA by p and the angle it makes with the axis of ξ by θ, then we have

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* Williamson's Diff. Calculus, p. 350, ex. 1.

† There is another movement whose point loci are limaçon. Its centrodes are a circle of radius r fixed, and another of radius $2r$ rolling upon and enclosing it. All the real double points of the curves are upon the circumference of the fixed centrode, whereas, for the movement under consideration, they cover the whole of the plane. Another distinction is as follows: If we regard the limaçon as obtained by extending the radii-vectores of the circle $\rho = 2p \cos\theta$ by a constant quantity m, then when the centrodes are equal circles, m is constant and equal to their radii for all the curves and p is equal to twice the distance of the tracing-point from the center of the moving centrode, while for the movement with unequal centrodes, $p = r$ and m is the variable distance from the tracing-point to the center of the moving centrode. This last movement is discussed by Cayley, "On the Kinematics of a Plane," Quarterly Journal, XVI, 1878, pp. 1–8, and by Schell, "Theorie der Bewegung und der Kräfte, I, Leipzig, 1879, p. 227.

$$\begin{aligned} \xi &= p \cos\theta - d \sin\theta, \\ \eta &= p \sin\theta + d \cos\theta. \end{aligned}$$

Substituting these values of ξ and η in (1) we obtain the equation of the locus of a point of l at the distance d from A, but it is the orthogonal projection of the curve in the plane $z = \frac{d}{s}$ described by that point of l_1 whose projection is P. Hence, the equation of the locus of l_1 is found by eliminating ξ, η and d from (1) by means of the above equations and $d = sz$. The result is

$$\begin{aligned} (x^2 + y^2)^2 - 2\,(p^2 + s^2z^2)(x^2 + y^2) - 4a^2x^2 + 4\,(c^2 - a^2)\,y^2 \\ - 8a^2x\,(p\cos\theta - sz\sin\theta) - 8\,(c^2 - a^2)\,y\,(p\sin\theta + sz\cos\theta) \\ + (p^2 + s^2z^2)^2 - 4a^2\,(p\cos\theta - sz\sin\theta)^2 \\ + 4\,(c^2 - a^2)(p\sin\theta + d\cos\theta)^2 = 0, \end{aligned} \tag{3}$$

a surface of the fourth order. It has a nodal circle of infinite radius in the infinitely distant plane that is parallel to α', for the point at infinity upon l describes a double circle. Since the whole of the double circle is actually described, the nodal circle of the surface lies upon it throughout its whole extent. The two pinch-points which are always situated upon this conic are imaginary.* Applying the above transformation to the equations $x = -\xi$, $y = \eta$ defining the position of the real double point of (1), we have

$$\begin{aligned} x &= -p\cos\theta + sz\sin\theta, \\ y &= p\sin\theta + sz\cos\theta, \end{aligned}$$

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* The most important articles upon the classification quartic scrolls are: Cremona, Mem. della R. Istoria di Bologna, Series II, T. VIII. Cayley, Philos. Trans., 1864, p. 559, and 1869, p. 111. Salmon, Geometry of Three Dimensions, Chap. XVI. Holgate, Amer. Journal, vol. XV, 1893, p. 344. The classifications of the first three authors, except for the order followed, are practically identical. Of the twelve species they recognize, Holgate has treated eight, describing under each a number of subforms. The place in the classifications of Cayley and Holgate of each of the surfaces to be described in this and the following section will be noted.

the equations of a straight line which is a nodal line of the surface. It makes the same angle with α as a generator.

The following types of surfaces will be distinguished:

I. The projection of the generatrix intersects the centrode C in two real and distinct points.

II. The projection of the generatrix is tangent to the centrode.

III. The projection of the generatrix intersects the centrode in two imaginary points.

Those points of l that are without C describe curves whose real double points are nodes, hence the corresponding portion of the nodal straight line lies upon the surface. This portion includes the point at infinity which is the point of intersection of the nodal straight line and the nodal circle. Assuming the conditions for type I, the points of l_1 whose projections are the points of intersection of l and C, describe curves having each a real cusp. The cusps mark pinch-points on the surface. Between them the sections of the surface parallel to α', corresponding to the points of l within C, have each a real conjugate point, showing the nodal line to be isolated. These scrolls are included under Cayley's seventh species and Holgate's F_3^4 subform 3. As l is brought nearer to the position of tangency with C, their two real points of intersection approach, which indicates for the surface generated by l_1 that the pinch-points of the nodal straight line approach each other, shortening its isolated segment until it finally vanishes, giving a surface of type II. These are of Cayley's eleventh species and Holgate's F_4^4. The nodal straight line lies entirely upon the surfaces of type III, its two pinch-points being imaginary. They are hence of Cayley's seventh species and Holgate's F_3^4 subform 5.

The general theory of quartic scrolls having a nodal straight line and nodal conic shows that the former is a generator when its pinch-points coincide (type II), and that when they are separated it is not a generator (types I and III).* This is readily verified in the present instance. From the equations $x = -\xi$, $y = \eta$ connecting the coordinates of the tracing point (ξ, η) in α with those of the double point (x, y) of its locus in α', it follows that the projection of the nodal straight line upon α' occupies the same relative position to C' that l does to C. In other words, by revolving α about a common tangent to C and

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*** In addition to the references cited see Holgate, Bulletin New York Math. Soc., III, 1894, p. 234.**

C' through 180°, the centrodes C and C' are brought into coincidence, and also l and the projection of the nodal straight line. The generatrix l_1 has (without loss of generality) been taken so as to pass through A, the foot of the perpendicular from O upon l, hence the nodal straight line passes through A', the foot of the perpendicular from O' upon its projection in α'. Now, in order that a position of l_1 can coincide with the nodal line, A must fall upon A' and l upon the projection of the nodal line. A moment's consideration convinces one that this can only occur under the conditions for type II. Assume it, and that the projections of the nodal line and generatrix coincide, forming the common tangent of the two centrodes. We have now the nodal line and a generator both passing through the pinch-point whose projection is the point of tangency. As previously remarked, they make the same angle with α', and if they do not coincide there are three sheets of the surface passing through the pinch-point, which is impossible.

Collecting the results obtained, we have the following: *When the centrodes of a plane movement are congruent ellipses tangent at homologous points, the locus of a carried straight line (which is neither parallel nor perpendicular to the plane of the centrodes) is a quartic scroll having a nodal straight line and a nodal circle intersecting it. The latter is of infinite radius and in the plane at infinity. It lies entirely upon the surface. If the projection of the generatrix upon the moving plane intersects the moving centrode in two real and distinct points (type I), the nodal straight line consists of an isolated segment and one lying upon the surface. The latter contains the point at infinity which is its intersection with the nodal circle. Two pinch-points bound the segments. If the projection of the generatrix is tangent to the centrode (type II), the nodal right line lies entirely upon the surface and has upon it a real double pinch-point. If the projection of the generatrix does not intersect the centrode (type III), the nodal straight line lies entirely upon the surface, and the two pinch-points upon it are imaginary. The nodal straight line is a generator in surfaces of type II but not in the others.*

It remains to note a few special varieties of these surfaces. The only ones having planes of symmetry are those whose generatrices project upon the axes of C. When l is the transverse axis, $y = 0$ is the plane of symmetry. The surface has two notable sections parallel to α', described by the points that project into the foci of C. These consist of a circle of radius $2a$ and an isolated point. The center of the circle and the isolated point projected orthogonally upon α'

are the foci of C'. A surface has one such section if l passes through one focus of C. In all surfaces having l passing through O, the mid-section between the pinch-points and parallel to α' is the doubly symmetric curve (2). When l is the conjugate axis, $x = 0$ is the plane of symmetry.

When the centrodes are circles, two generatrices making the same angle with α and whose projections pass the same distance from O, give congruent surfaces. If l passes through O, the surface has a plane of symmetry perpendicular to α' and passing through O' and a single circular section parallel to α'.

§2.—*The Centrodes are Hyperbolas.*

In this section the same notation as in the preceding will be used. The equations of the point and line loci hold good, subject to the condition $c > a$. This movement is also a degenerate case of three-bar motion,* but the fixed link is one of the longer sides of the jointed parallelogram instead of one of the shorter.

The locus of the point (ξ, η) of the plane α is a unicursal quartic having nodes at the circular points at infinity and a real double point at $(-\xi, \eta)$. The latter is a node, cusp or conjugate point according as (ξ, η) is without (on the convex side of), on or within the centrode C, and it falls upon α' in a corresponding position relative to C'.

The curves having a real node consist of two closed loops, mutually exterior, united at the node. They are symmetrical with respect to the transverse or conjugate axis of C' for the tracing point upon the corresponding axis of C. The locus of O, equation (2), is symmetrical with respect to both axes and resembles the lemniscate, which it in fact is, for $c = \sqrt{2}a$.†

As the tracing point approaches the centrode from without, one loop of the curve shrinks to a cusp as the centrode is reached. The curve then consists of a single closed loop with a cusp protruding outwards. The tangent to the cusp is normal to C' at the cusp.

A description of the manner in which the centrodes roll enables one to gain an insight into the nature of the paths of distant points of α. Starting with two vertices of C and C' in contact as C rolls, the point of contact recedes toward infinity and two asymptotes approach coincidence. The other branches of C and C' are not tangent, but as the asymptotes pass the position of coincidence they become tangent. At the same time the original pair cease to touch. The point of contact now approaches from infinity in the opposite direction to that in which it receded. It finally reaches the vertex opposite to that at which it started, thus completing a half cycle of the movement. Each branch of C rolls only upon a corresponding branch of C'. The movement is the exact analogue of that for the elliptical centrodes; in a complete cycle the point of contact travels once around the curve, and to each point of the fixed centrode corresponds a point of the moving one.

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* Burmester, Kinematik, I, pp. 802–804.

† Haedenkamp, Archiv für Math. u. Phys., III, 1843, p. 400.

The result of the movement is to make α rotate (about a point remaining within a finite area of α') from a position of coincidence of two asymptotes to the next such coincidence through an angle equal to twice the angle (that includes a branch) between the asymptotes of a centrode. The next half cycle rotates the plane in the opposite direction to its original position. The locus of a distant point of α must resemble the arc of a circle twice described, which it becomes in the limit for a tracing point at infinity. As equation (1) cannot define a portion of a circle, it gives the whole circumference. Hence a surface generated by a carried straight line has its nodal circle in part upon it and in part isolated.

The curve, by a point within a branch of C, has a conjugate point within the corresponding branch of C', and consists in addition of a closed curve lying without that branch.

The character of the surfaces generated by a carried line can be now readily inferred from the given properties of the point loci as in the preceding section. It will be necessary only to state here the results, which are as follows: *When the centrodes of a plane movement are congruent hyperbolas tangent at homologous points, the locus of a carried straight line (which is neither parallel nor perpendicular to the plane of the centrodes) is a quartic scroll having a nodal straight line and a nodal circle, the two intersecting. The latter is of infinite radius in the plane at infinity. It consists of two segments bounded by pinch-points, one segment lying upon the surface, the other isolated. There are six types of surfaces distinguished by the position of the projection l of the generatrix with respect to the moving centrode C.*

I. l intersects both branches of C in real finite points. The nodal straight line consists of two segments bounded by pinch-points. One is isolated and contains the point at infinity, the other lies upon the surface. The intersection of the two nodal lines is upon the isolated segments of both.

II. l intersects one branch of C in two real finite points. The nodal straight line has two segments separated by pinch-points. The segment lying upon the surface contains the point at infinity which is its intersection with the nodal circle. The point of intersection falls in that segment of the nodal circle that lies upon the surface.

III. l is tangent to C but is not an asymptote. The nodal straight line lies wholly in the surface, and has upon it a double pinch-point. The intersection of the nodal lines is in that segment of the nodal circle that lies upon the surface.

IV. l intersects C in imaginary points. The nodal straight line lies wholly upon the surface, its two pinch-points being imaginary. The intersection of the nodal lines is in that segment of the circle that lies upon the surface.

V. l is parallel to an asymptote of C. The nodal straight line has two segments with two pinch-points bounding them. One of these pinch-points and one upon the nodal circle coincide with the intersection of the nodal lines.

VI. l is an asymptote of C. The nodal straight line lies wholly upon the surface and has a double pinch-point at infinity. This double pinch-point and a pinch-point of the nodal circle coincide with the intersection of the nodal lines.

The nodal straight line in types III and VI is a generator, but in the other types it is not.

The types III and VI belong to Cayley's eleventh species and Holgate's F_4^4. The remaining types are included in Cayley's seventh species. Types I and II belong to Holgate's F_3^4, subform 1; type IV to F_3^4, subform 2; type V to F_3^4, subform 4.*

If l is the transverse axis of C, the surface is symmetrical with respect to $y=0$, and it has two circular sections parallel to α' and the mid-section (2). When l passes through a single focus, the surface has one circular section. For l coincident with the conjugate axis of C, the surface is symmetrical with respect to $x=0$.

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* It is interesting to note that the movements of this and the preceding section furnish a means of mechanically generating examples of all the "subforms"—as recognized by Holgate, Amer. Jour., vol. XV—of scrolls of the fourth order, leaving a nodal conic and nodal straight line.

§3.—*The Centrodes are Parabolas.*

The centrodes are congruent parabolas so placed that, by rolling, their vertices can be brought into coincidence. Let the centrode in α' as before be denoted by C', its focus, vertex and directrix by F', O' and d' respectively, and let the corresponding elements for the other centrode, C, be F, O and d. For axes of x, y and z, take respectively the tangent at the vertex of C', its axis and a perpendicular to α' through O'; and for coordinate axes carried by α take the tangent at the vertex of C for axes of ξ and its axes for axes of η. The positive halves of the axes of y and η are selected so as to pass through F' and F.

It is well known that the focus of each centrode describes the directrix of the other, or conversely, the directrix of each passes through the focus of the other.[*] Hence, the movement can also be obtained by taking a point F' and a straight line d' in α' and a congruent configuration F, d in α and then conditioning F to remain upon d' and d to pass through F'.[†] As thus defined, the movement is, however, more general than the preceding. Like the three-bar equivalents of the movements of sections 1 and 2, it is a degenerate case of one of higher order. For it is possible to bring d parallel to d' and have them remain so while α is being translated in their direction. The locus of any point of α is then a straight line parallel to d and d'. Such a straight line with the curve of the third order traced by the same point when the centrodes are the parabolas C and C' constitute a degenerate curve of the fourth order. If the distance from d' to F' is not equal to that from d to F, a point of α in general describes a non-degenerate curve of the fourth order.[‡] These curves have been studied by Roberts.[||]

His paper also treats the cubics resulting by degeneration when the distance from F' to d' is equal to that from F to d, i. e. when the centrodes are congruent parabolas. We proceed to give such of their properties as will be required in describing the surfaces generated by a carried line. The equation of the locus of the point (ξ, η) is

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[*] Burmester, Kinematik, I, p. 884.

[†] The middle point O of the perpendicular from F to d describes the cissoid of Diocles. The first known mention of this method of describing it is in Newton's Arithmetica universalis. See A. v. Braunmühl, "Studie über Curvenerzeugung," in the "Katalog mathematischer Modelle," by Dyck.

[‡] When F is upon d we have the mechanism of Nicomedes for drawing the conchoid. Braunmühl in the "Katalog mathematischer Modelle," by Dyck, p. 57.

[||] Proc. Lond. Math. Soc., II, 1869, pp. 125–136.

$$y^3 + x^2y - \eta y^2 + (a - \eta)x^2 - (\xi^2 + \eta^2)y - 2a\xi x + \eta^3 + \eta\xi^2 + a\xi^2 = 0, \quad (4)$$

where a is the distance from the focus to the vertex of the centrodes. The curve is unicursal, having a double point at (ξ, η), which is a node, cusp or conjugate point according as the tracing point is without, on or within C. In α' nodes fall without, cusps upon, and conjugate points within C'. The line $y = \eta - a$ is an asymptote which the curve crosses at the point $\left(\frac{2a\eta + 2\xi^2 - a^2}{2\xi}, \eta - a\right)$. The abscissa of this point is infinite for $\xi = 0$, i. e. for all the curves whose tracing-points are upon the axis of C. The point of contact of the asymptote is an inflexion, and the curves are symmetrical with respect to $x = 0$. The only exception to the condition $\xi = 0$ for an inflexion upon the asymptote is when in addition $\eta = \frac{a}{2}$. The tracing-point is then the focus of C and its locus degenerates to the focus and directrix of C'. The vertex of C, as has been remarked above, is the cissoid.

The equation of the surface generated by a straight line carried by α is obtained by the method of section 1. As there, the generatrix is assumed to be neither perpendicular nor parallel to α. The former would give right cubical cylinders upon (4) as bases, the latter a plane curve. Let s be the cotangent of the angle the generatrix l_1 makes with α, l its projection, and A the foot of the perpendicular from O to l. Denoting the length of OA by p and its angle with the axis of ξ by θ, we have for the coordinates of the point of l distant d from A

$$\xi = p\cos\theta - d\sin\theta,$$
$$\eta = p\sin\theta + d\cos\theta.$$

Eliminating ξ, η and d from (4) by means of these and $d = sz$, we have for the equation of the locus of l_1

$$\begin{aligned} y^3 + x^2y - (p\sin\theta + sz\cos\theta)\,y^2 + (a - p\sin\theta + sz\cos\theta)\,x^2 \\ - (p^2 + s^2z^2)(y - p\sin\theta - sz\cos\theta) - 2ax\,(p\cos\theta - sz\sin\theta) \\ + a\,(p\cos\theta - sz\sin\theta)^2 = 0. \end{aligned}$$

It is of the third order,* and has a nodal straight line whose equations are

$$x = p\cos\theta - sz\sin\theta,$$
$$y = p\sin\theta + sz\cos\theta.$$

obtained by applying the above transformation to the equations $x = \xi$, $y = \eta$, which define the double point of (4).

After what has been given upon the movement with elliptical centrodes, the following summary of results may be given without further explanation: *When the centrodes of a plane movement are congruent parabolas tangent at homologous points, the locus of a carried line (which is neither perpendicular nor parallel to the plane of the centrodes) is a cubic scroll having a nodal straight line. They are of four types according to the position of the projection l of the generatrix with respect to the moving centrode C.*

I. l intersects C in two real finite points. The nodal line consists of two segments bounded by pinch-points. One segment lies upon the surface and contains the point at infinity, the other is isolated.

II. l is tangent to C. The nodal line lies wholly upon the surface, and has upon it a double pinch-point.

III. l does not intersect C in real points. The nodal line lies wholly upon the surface and its two pinch-points are imaginary.

IV. l is parallel to the axis of C. The nodal line has two segments bounded by pinch-points. One segment lies upon the surface, the other is isolated. Both extend to infinity, since one of the pinch-points is at infinity.

The surfaces for which l is $\xi = 0$ have $x = 0$ a plane of orthogonal symmetry, and are the only ones having such a plane. If l passes through the vertex of C, the section of the surface through the corresponding pinch-point and parallel to α' is the cissoid. If l passes through the focus, one section parallel to α' is a straight line with an isolated point. The first surface mentioned possesses both of these sections.

BROOKLYN, N. Y., *August* 8, 1898.

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* The order of the surface generated by a carried straight line is not always as low as the apparent order of the point loci of the same movement. Thus the ellipsograph (trammel) gives curves of the second order and surfaces of the fourth. From the point of view of the latter, the ellipses drawn must be regarded as of the fourth order and include the line at infinity taken twice.

Quinquisection of the Cylotomic Equation.

By J. C. Glashan, *Ottawa, Canada.*

(Read in abstract before Section A of the British Association for the Advancement of Science, August 24, 1897.)

Let p be a prime number $= 5n + 1$, t be a primitive root of p and

$$(\chi^p - 1)/(\chi - 1) = 0, \tag{i}$$

also let

$$\left.\begin{aligned} \eta_0 &= \chi + \chi^{t^5} + \chi^{t^{10}} + \chi^{t^{15}} + \dots, \\ \eta_1 &= \chi^{t} + \chi^{t^6} + \chi^{t^{11}} + \chi^{t^{16}} + \dots, \\ \eta_2 &= \chi^{t^2} + \chi^{t^7} + \chi^{t^{12}} + \chi^{t^{17}} + \dots, \\ \eta_3 &= \chi^{t^3} + \chi^{t^8} + \chi^{t^{13}} + \chi^{t^{18}} + \dots, \\ \eta_4 &= \chi^{t^4} + \chi^{t^9} + \chi^{t^{14}} + \chi^{t^{19}} + \dots \end{aligned}\right\} \tag{ii}$$

and

$$\left.\begin{aligned} \eta_0\eta_1 &= \alpha\eta_0 + \gamma\eta_1 + b\eta_2 + c\eta_3 + d\eta_4, \\ \eta_0\eta_2 &= \beta\eta_0 + e\eta_1 + \delta\eta_2 + f\eta_3 + g\eta_4, \end{aligned}\right\} \tag{iii}$$

$$\therefore \quad \eta_0 + \eta_1 + \eta_2 + \eta_3 + \eta_4 + 1 = 0. \tag{iv}$$

Making $\chi = 1$ in equations (iii) and dividing through by n,

$$\left.\begin{aligned} \alpha + \gamma + b + c + d &= n, \\ \beta + e + \delta + f + g &= n. \end{aligned}\right\} \tag{v}$$

On substituting χ^t, χ^{t^2}, χ^{t^3}, χ^{t^4} successively for χ, we obtain the eight equations which may be formed from (iii) by the substitutions $(\eta_0\eta_1\eta_2\eta_3\eta_4)^m$, $m = 1, 2, 3, 4$. These, with the two equations of (iii), form the group of ten equations

$$\left.\begin{aligned} &\eta_m\eta_{m+1} = \alpha\eta_m + \gamma\eta_{m+1} + b\eta_{m+2} + c\eta_{m+3} + d\eta_{m+4}, \\ &\eta_m\eta_{m+2} = \beta\eta_m + e\eta_{m+1} + \delta\eta_{m+2} + f\eta_{m+3} + g\eta_{m+4}, \\ &\text{in which} \\ &\eta_{m+5} = \eta_m \\ &\text{and} \\ &m = 0,\ 1,\ 2,\ 3,\ 4. \end{aligned}\right\} \tag{vi}$$

From (vi) and (iv),

$$\left.\begin{aligned}(\eta_0\eta_1)\,\eta_2 &= -b\eta_2 + (\alpha - b)\,\eta_0\eta_2 + (\gamma - b)\,\eta_1\eta_2 + (c - b)\,\eta_2\eta_3 + (d - b)\,\eta_2\eta_4,\\ = (\eta_1\eta_2)\,\eta_0 &= -d\eta_0 + (\alpha - d)\,\eta_0\eta_1 + (\gamma - d)\,\eta_0\eta_2 + (b - d)\,\eta_0\eta_3 + (c - d)\,\eta_0\eta_4,\\ = (\eta_0\eta_2)\,\eta_1 &= -e\eta_1 + (\beta - e)\,\eta_0\eta_1 + (\delta - e)\,\eta_1\eta_2 + (f - e)\,\eta_1\eta_3 + (g - e)\,\eta_1\eta_4.\end{aligned}\right\}\text{(vii)}$$

On making $\chi = 1$ in the right-hand members of these equations, we obtain

$$\begin{aligned}(\alpha + \gamma + c + d - 4b)\,n^2 - bn &= (\alpha + \gamma + b + c - 4d)\,n^2 - dn\\ &= (\beta + \delta + f + g - 4e)\,n^2 - en;\\ \therefore\quad (n - 5b)\,n - b &= (n - 5d)\,n - d = (n - 5e)\,n - e;\\ \therefore\quad b &= d = e. \qquad\qquad \text{(viii)}\end{aligned}$$

Similarly, from $(\eta_0\eta_1)\,\eta_3 = (\eta_1\eta_3)\,\eta_0 = (\eta_0\eta_3)\,\eta_1$, we obtain

$$c = f = g. \qquad\qquad \text{(ix)}$$

On writing b for both d and e, and c for both f and g, and substituting for the products $\eta_0\eta_1$, $\eta_0\eta_2$, ..., $\eta_3\eta_4$ in the right-hand members of the equations (vii) the values of these products as given in (vi), we obtain

$$b = (\alpha - b)(\alpha - \beta) + (\gamma - b)(\beta - b) + (c - b)(\gamma - c). \qquad\qquad \text{(x)}$$

Similarly, from $(\eta_0\eta_2)\,\eta_4 = (\eta_2\eta_4)\,\eta_0 = (\eta_0\eta_4)\,\eta_2$, we obtain

$$c = (\beta - c)(\beta - \gamma) + (\delta - c)(\gamma - c) + (b - c)(\delta - b). \qquad\qquad \text{(xi)}$$

It is worthy of notice that (xi) may be obtained from (x) by the substitution $\{(\alpha\beta\gamma\delta)(bc)\}$. This is a consequence of the substitution $\{(\eta_1\eta_2\eta_4\eta_3)(\alpha\beta\gamma\delta)(bc)\}$, leaving the equation-group (vi) unchanged.

The eight equations of (v), (viii), (ix), (x) and (xi) may be written under the form

$$\left.\begin{aligned}&b = d = e,\\ &c = f = g,\\ &\alpha + \gamma = n - 2b - c,\\ &\beta + \delta = n - b - 2c,\\ &\alpha\gamma + \beta\delta = n^2 - (4n + 1)(b + c) + 5b^2 + 7bc + 5c^2,\\ &5\alpha^2\gamma^2 - \alpha\gamma\,[5\,\{n^2 - n\,(5b + 3c) + 7b^2 + 7bc + 3c^2\} - 6b - 4c]\\ &\qquad + n^4 - 5n^3\,(2b + c) - n^2\,(40b^2 + 35bc + 10c^2 - 2b - c)\\ &\qquad - n\,(75b^3 + 90b^2c + 40bc^2 + 10c^3 - 11b^2 - 7bc - 2c^2)\\ &\qquad + 5\,(11b^4 + 17b^3c + 9b^2c^2 + 3bc^3 + c^4)\\ &\qquad - (16b^3 + 15b^2c + 3bc^2 + c^3) + 2b^2 = 0.\end{aligned}\right\}\text{(xii)}$$

The last of these equations may also be written under the form

$$4pA^2 = (B + 5p)^2 - 5D^2$$

in which

$$A = 25(b + c) - 2(p + 1),$$
$$B = 25\{(b + c)^2 + 5bc - (p + 1)(b + c)\} + p^2 + p + 1,$$
$$D = 50\alpha\gamma - (p - 1)^2 + 5(p - 1)(5b + 3c) - 25(7b^2 + 7bc + 3c^2) + 10(3b + 2c).$$

Computing the values of the symmetric functions $\Sigma\eta_0\eta_1$, $\Sigma\eta_0\eta_1\eta_2$, etc., in terms of the coefficients α, β, γ, δ, b, c, the quintic whose roots are η_0, η_1, η_2, η_3, η_4 is found to be

$$\begin{aligned}\eta^5 + \eta^4 - 2n\eta^3 - \{p(b + c) - 2n^2\}\eta^2 \\ + [p\{n(b + c) - b^2 - 3bc - c^2\} - n^3]\eta \\ + (2b^2 - \alpha\gamma)(2c^2 - \beta\delta) - (b + c)^4 + n(b + c)^3 + bc(b + c)^2 \\ - (3n + 1)bc(b + c) + b^2c^2 = 0.\end{aligned} \qquad \text{(xiv)}$$

If K denote the constant term of this equation

$$\begin{aligned}5K = p\{(b - c)\alpha\gamma - n^2(2b + c) + n(2b + c)(3b + c) \\ - (6b^3 + 10b^2c + 8bc^2 + c^3) + b^2\} + n^4.\end{aligned} \qquad \text{(xv)}$$

If (xiv) be transformed by substituting $\frac{1}{5}(z - 1)$ for η, it becomes

$$\begin{aligned}z^5 - 10pz^3 - 5p\{25(b + c) - 2(p + 1)\}z^2 \\ - 5p[25\{(b + c)^2 + 5bc - (p + 1)(b + c) + p^2 + p + 1]z \\ + 3125\{(2b^2 - \alpha\gamma)(2c^2 - \beta\delta) - (b + c)^4\} \\ + 125\{5(b + c)^2 + 2bc\}(b + c)(p - 1) \\ + 125(5bc + p)\{5(b + c)^2 + 5bc - p(b + c)\} + 5p(p^2 - p + 1) + 1 = 0.\end{aligned} \qquad \text{(xvi)}$$

Writing this equation

$$z^5 - 10pz^3 - 5pAz^2 - 5pBz - pC = 0$$

it will be found that

$$\left.\begin{aligned}4(B + p) = A^2 - 125(b - c)^2, \\ \{2C - A(B - p)\}^2 = 125(b - c)^2\{(B + 5p)^2 - 4pA^2\}.\end{aligned}\right\} \qquad \text{(xvii)}$$

To solve equation (xiv), let

$$(\theta^5 - 1)/(\theta - 1) = 0,$$

and $$\eta_m = \tfrac{1}{5}(y_0 + y_1\theta^{-m} + y_2\theta^{-2m} + y_3\theta^{2m} + y_4\theta^{m}),$$

and ∴ $$y_m = \eta_0 + \eta_1\theta^{m} + \eta_2\theta^{2m} + \eta_3\theta^{3m} + \eta_4\theta^{4m},$$

$$m = 0,\ 1,\ 2,\ 3,\ 4$$

then will

$$\left.\begin{aligned}&y_0 = -1,\\&y_1y_4 = y_2y_3 = p,\\&y_1^2y_3 + y_2^2y_1 + y_3^2y_4 + y_4^2y_2 = pA,\\&y_1^3y_2 + y_2^3y_4 + y_3^3y_1 + y_4^3y_3 = p\,(B+p),\\&y_1^5 + y_2^5 + y_3^5 + y_4^5 = pC,\end{aligned}\right\}\qquad\text{(xviii)}$$

$$\left.\begin{aligned}\therefore\ &y_1^2y_3 + y_4^2y_2 = \tfrac{1}{2}p\,\{A + 5\,(b-c)\sqrt{5}\} = \tfrac{1}{2}px,\\&y_2^2y_1 + y_3^2y_4 = \tfrac{1}{2}p\,\{A - 5\,(b-c)\sqrt{5}\} = \tfrac{1}{2}p\lambda,\\&y_1^2y_3 - y_4^2y_2 = \tfrac{1}{2}p\sqrt{(x^2-16p)} = \tfrac{1}{2}p\mu,\\&y_3^2y_4 - y_2^2y_1 = \tfrac{1}{2}p\sqrt{(\lambda^2-16p)} = \tfrac{1}{2}p\nu,\end{aligned}\right\}\qquad\text{(xix)}$$

$$\left.\begin{aligned}\therefore\ &y_1 = \tfrac{1}{2}\sqrt[5]{\{\tfrac{1}{2}p\,(x+\mu)^2(\lambda-\nu)\}},\\&y_2 = \tfrac{1}{2}\sqrt[5]{\{\tfrac{1}{2}p\,(x-\mu)\,(\lambda-\nu)^2\}},\\&y_3 = \tfrac{1}{2}\sqrt[5]{\{\tfrac{1}{2}p\,(x+\mu)\,(\lambda+\nu)^2\}},\\&y_4 = \tfrac{1}{2}\sqrt[5]{\{\tfrac{1}{2}p\,(x-\mu)^2(\lambda+\nu)\}},\end{aligned}\right\}\qquad\text{(xx)}$$

and
$$\eta_m = \tfrac{1}{5}\,(y_0 + y_1\theta^{-m} + y_2\theta^{-2m} + y_3\theta^{2m} + y_4\theta^{m}).\qquad\text{(xxi)}$$

In the Proceedings of the London Mathematical Society, vol. XII, p. 16, the late Professor Cayley has given tables of the values of the coefficients α, β, γ, δ, b, c, d, e, f, g and of the coefficients of the quintic in η for all values of the prime $p = 5n + 1$ up to $p = 71$, and on page 63 of vol. XVI of the Proceedings he has given from Legendre's "Théorie des Nombres," 3e edition, t. II, p. 213, the quintic in η for $p = 641$.* See also Cayley's Collected Mathematical Papers, vol. XI, p. 316, and vol. XII, p. 73. The following tables extend these values to $p = 641$. The values in table 1 are subject to the substitutions $\{(\alpha\beta\gamma\delta)(bgdf)(ce)\}^m$.

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* In Cayley's Table 2, in the equation for $p = 31$, coefficient of η^2, for -2 read -21, and in the equation for $p = 61$, coefficient of η^0, for -26 read -13. In the equation from Legendre, for $+5238$ read $+5328$.

TABLE I.

VALUES OF

p	a β	γ e	b δ	c f	d g	p	a β	γ e	b δ	c f	d g
11	1 0	0 0	0 0	1 1	0 1	281	14 8	10 9	9 11	14 14	9 14
31	1 2	0 2	2 0	1 1	2 1	311	10 14	17 12	12 14	11 11	12 11
41	2 3	2 1	1 0	2 2	1 2	331	14 10	9 14	14 12	15 15	14 15
61	3 0	2 2	2 4	3 3	2 3	401	15 22	18 16	16 12	15 15	16 15
71	2 5	4 3	3 2	2 2	3 2	421	19 20	18 14	14 12	19 19	14 19
101	3 6	6 4	4 4	3 3	4 3	431	20 12	14 16	16 18	20 20	16 20
131	5 2	4 6	6 8	5 5	6 5	461	16 24	18 21	21 15	16 16	21 16
151	8 6	6 4	4 4	8 8	4 8	491	18 26	21 20	20 14	19 19	20 19
181	8 3	6 7	7 10	8 8	7 8	521	24 15	18 19	19 22	24 24	19 24
191	8 8	4 9	9 5	8 8	9 8	541	18 23	28 21	21 24	20 20	21 20
211	6 9	8 11	11 10	6 6	11 6	571	24 18	26 20	20 28	24 24	20 24
241	12 6	8 8	8 10	12 12	8 12	601	23 24	30 22	22 28	23 23	22 23
251	8 14	10 12	12 8	8 8	12 8	631	26 25	30 23	23 30	24 24	23 24
271	10 6	9 12	12 14	11 11	12 11	641	28 18	24 24	24 30	28 28	24 28

TABLE II.

COEFFICIENTS OF THE QUINTIC EQUATIONS.

p	η^5	η^4	η^3	η^2	η^1	η^0
11	1	1	− 4	− 3	+ 3	+ 1
31	1	1	− 12	− 21	+ 1	+ 5
41	1	1	− 16	+ 5	+ 21	− 9
61	1	1	− 24	− 17	+ 41	− 13
71	1	1	− 28	+ 37	+ 25	− 1
101	1	1	− 40	+ 93	− 21	− 17
131	1	1	− 52	− 89	+ 109	+ 193
151	1	1	− 60	− 12	+ 784	+ 128
181	1	1	− 72	− 123	+ 223	− 49
191	1	1	− 76	− 359	− 437	− 155
211	1	1	− 84	− 59	+ 1661	+ 269
241	1	1	− 96	− 212	+ 1232	+ 512
251	1	1	− 100	− 20	+ 1504	+ 1024
271	1	1	− 108	− 401	− 13	+ 845
281	1	1	− 112	− 191	+ 2257	+ 967
311	1	1	− 124	+ 535	− 413	− 539
331	1	1	− 132	− 887	− 1843	− 1027
401	1	1	− 160	+ 369	+ 879	− 29
421	1	1	− 168	+ 219	+ 3853	− 3517
431	1	1	− 172	− 724	+ 1824	+ 1728
461	1	1	− 184	− 129	+ 4551	+ 5419
491	1	1	− 196	+ 59	+ 2019	+ 1377
521	1	1	− 208	− 771	+ 4143	+ 2083
541	1	1	− 216	+ 1147	− 805	− 2629
571	1	1	− 228	+ 868	+ 3056	− 7552
601	1	1	− 240	+ 1755	− 3731	+ 2399
631	1	1	− 252	+ 2095	− 5735	+ 5069
641	1	1	− 256	− 564	+ 5328	− 5120
	a	$5b$	$10c$	$10d$	$5e$	f

$$af - 3be + 2cd \equiv 0 \text{ mod. } p,$$
$$ae - 4bd + 3c^2 \equiv 0 \quad \text{“}$$
$$bf - 4ce + 3d^2 \equiv 0 \quad \text{“}$$

On the m-Fold Section of the Cyclotomic Equation in the Case of m Prime.

By J. C. Glashan, *Ottawa, Canada.*

Let m be any odd prime number, p a prime $= mn + 1$, t a primitive root of p,

$$(\chi^p - 1)/(\chi - 1) = 0 \tag{i}$$

and

$$\eta_h = \chi^{t^h} + \chi^{t^{h+m}} + \chi^{t^{h+2m}} + \ldots\ldots + \chi^{t^{h+(n-1)m}}, \tag{ii}$$

$$h = 0,\ 1,\ 2,\ \ldots\ldots,\ m-1$$

then will

$$1 + \eta_0 + \eta_1 + \eta_2 + \eta_3 + \ldots\ldots + \eta_{m-1} = 0. \tag{iii}$$

Let

$$\eta_0\eta_r = e_{r,\,0}\eta_0 + e_{r,\,1}\eta_1 + e_{r,\,2}\eta_2 + \ldots\ldots + e_{r,\,c}\eta_c + \ldots\ldots e_{r,\,m-1}\eta_{m-1} \tag{iv}$$

$$r = 1,\ 2,\ 3,\ \ldots\ldots,\ m-1.$$

On making $\chi = 1$ in (ii) and (iv), which may be done, r being > 0 and therefore (iii) not occurring in the reduction of $\eta_0\eta_r$ to the form $e_{r,\,0}\eta_0 + e_{r,\,1}\eta_1 + \ldots$, we obtain

$$e_{r,\,0} + e_{r,\,1} + e_{r,\,2} + \ldots\ldots + e_{r,\,m-1} = n. \tag{v}$$

Writing $m - r$ for r in (iv), it becomes

$$\begin{aligned}\eta_0\eta_{m-r} = e_{m-r,\,0}\eta_0 + e_{m-r,\,1}\eta_1 + \ldots\ldots + e_{m-r,\,c-r}\eta_{c-r} \\ + \ldots\ldots + e_{m-r,\,m+g-r}\eta_{m+g-r} + \ldots\ldots + e_{m-r,\,m-1}\eta_{m-1}\end{aligned} \tag{vi}$$

$$c \nless r, \quad g < r$$

But p being prime, $\chi^t, \chi^{t^2}, \ldots\ldots, \chi^{t^{p-2}}$ are prime roots of $\chi^p - 1 = 0$, and therefore the equations of form (iv) are all subject to the substitutions $(\eta_0\eta_1\eta_2 \ldots\ldots \eta_{m-1})^h$. Operating on (iv) with $(\eta_0\eta_1\eta_2 \ldots\ldots \eta_{m-1})^{m-r}$, it becomes

$$\begin{aligned}\eta_0\eta_{m-r} = e_{r,\,r}\eta_0 + e_{r,\,r+1}\eta_1 + e_{r,\,r+2}\eta_2 + \ldots\ldots + e_{r,\,c}\eta_{c-r} + \ldots\ldots \\ + e_{r,\,g}\eta_{m+g-r} + \ldots\ldots + e_{r,\,r-1}\eta_{m-1}.\end{aligned} \tag{vii}$$

Comparing equations (vi) and (vii) we obtain

$$\left.\begin{aligned} e_{m-r,\, m+c-r} &= e_{r,\, c} \text{ if } c < r, \\ e_{m-r,\, c-r} &= e_{r,\, c} \text{ if } c = \text{ or } > r. \end{aligned}\right\} \tag{viii}$$

From equations (iii) and (iv), we easily obtain

$$\begin{aligned} \eta_0\eta_r &= -e_{r,\, c} + (e_{r,\, 0} - e_{r,\, c})\,\eta_0 + (e_{r,\, 1} - e_{r,\, c})\,\eta_1 + \ldots\ldots, \\ \eta_0\eta_c &= -e_{c,\, r} + (e_{c,\, 0} - e_{c,\, r})\,\eta_0 + (e_{c,\, 1} - e_{c,\, r})\,\eta_1 + \ldots\ldots, \end{aligned}$$

$$\begin{aligned} \because\quad & -e_{r,\, c}\eta_c + (e_{r,\, 0} - e_{r,\, c})\,\eta_0\eta_c + (e_{r,\, 1} - e_{r,\, c})\,\eta_1\eta_c + \ldots\ldots \\ & = -e_{c,\, r}\eta_r + (e_{c,\, 0} - e_{c,\, r})\,\eta_0\eta_r + (e_{c,\, 1} - e_{c,\, r})\,\eta_1\eta_r + \ldots\ldots \end{aligned}$$

Making $\chi = 1$ in this equation, which may be done provided neither r nor c is zero, we obtain

$$e_{c,\, r} = e_{r,\, c}. \tag{ix}$$

If, now, we write

$$\eta_0^2 = (e_{0,\, 0} - n - 1)\,\eta_0 + (e_{0,\, 1} - n)\,\eta_1 + (e_{0,\, 2} - n)\,\eta_2 + \ldots\ldots + (e_{0,\, m-1} - n)\,\eta_{m-1},$$

$$e_{m+h} = e_h \text{ and } e_{-h} = e_{m-h},$$

equations (v), (viii) and (ix) become

$$\left.\begin{aligned} & e_{r,\, 0} + e_{r,\, 1} + e_{r,\, 2} + \ldots\ldots + e_{r,\, m-1} = n, \\ & e_{r,\, c} = e_{c,\, r} = e_{m-r,\, c-r} = e_{c-r,\, m-r} = e_{m-c+r,\, m-c} = e_{m-c,\, m-c+r}, \\ & \qquad r = 0,\ 1,\ 2,\ 3,\ \ldots\ldots,\ m-1 \\ & \qquad c = 0,\ 1,\ 2,\ 3,\ \ldots\ldots,\ m-1 \end{aligned}\right\} \tag{x}$$

Returning to equations (iii) and (iv),

$$\begin{aligned} (\eta_0\eta_r)\,\eta_c &= -e_{r,\, c}\eta_c + (e_{r,\, 0} - e_{r,\, c})\,\eta_0\eta_c + (e_{r,\, 1} - e_{r,\, c})\,\eta_1\eta_c + \ldots\ldots \\ = (\eta_r\eta_c)\,\eta_0 &= -e_{c-r,\, m-r}\eta_0 + (e_{c-r,\, m-r+1} - e_{c-r,\, m-r})\,\eta_0\eta_1 + \ldots\ldots \end{aligned}$$

Substituting for the products $\eta_0\eta_1,\ \eta_0\eta_2,\ \ldots\ldots,\ \eta_0\eta_c,\ \eta_1\eta_2,\ \ldots\ldots,$ their linear values as given in equation (iv)$(\eta_0\eta_1\eta_2 \ldots\ldots \eta_{m-1})^h$, we obtain

$$\begin{aligned} & (e_{r,\, 0} - e_{r,\, c})\, e_{c,\, 0} + (e_{r,\, 1} - e_{r,\, c})\, e_{c-1,\, m-1} + (e_{r,\, 2} - e_{r,\, c})\, e_{c-2,\, m-2} + \ldots\ldots \\ & = -e_{c-r,\, m-r} + (e_{c-r,\, m-r+1} - e_{c-r,\, m-r})\, e_{1,\, 0} + (e_{c-r,\, m-r+2} - e_{c-r,\, m-r})\, e_{2,\, 0} + \ldots\ldots \\ \therefore\quad & e_{r,\, c} + (e_{r,\, 0} - e_{r,\, c})\, e_{0,\, c} + (e_{r,\, 1} - e_{r,\, c})\, e_{1,\, c} + (e_{r,\, 2} - e_{r,\, c})\, e_{2,\, c} + \ldots\ldots \\ & = (e_{r-1,\, c-1} - e_{r,\, c})\, e_{1,\, 0} + (e_{r-2,\, c-2} - e_{r,\, c})\, e_{2,\, 0} + (e_{r-3,\, c-3} - e_{r,\, c})\, e_{3,\, 0} + \ldots\ldots \end{aligned} \tag{xi}$$

in which $c > r > 0$.

It may now be seen that if $m > 3$ and the equations $e_{r,\, c} = e_{c,\, r} = \ldots\ldots$ of (x) be considered to be identities and therefore be not counted as equations proper, the number of e-coefficients will be $\frac{1}{6}(m+1)(m+2)$, the number of independent linear equations connecting these will be $\frac{1}{2}(m+1)$, and the number of independent quadratic equations connecting them will be $\frac{1}{6}(m-1)(m-2)$. The coefficient $e_{0,0}$ occurs only in the equation

$$e_{0,0}+e_{0,1}+e_{0,2}+\ldots\ldots+e_{0,m-1}=n.$$

Omitting this equation and the coefficient $e_{0,0}$, there will remain $\frac{1}{6}(m-1)(m+4)$ distinct e-coefficients connected by $\frac{1}{2}(m-1)$ linear and $\frac{1}{6}(m-1)(m-2)$ quadratic equations. These e-coefficients may most conveniently be considered as forming a rectangular array of elements in $m-1$ rows and m columns such that

$$\begin{aligned} e_{r,c} &= e_{c-r,\,m-r}=e_{m-c,\,m-c+r} \\ &= e_{c,\,r}=e_{m-r,\,c-r}=e_{m-c+r,\,m-c}, \end{aligned}$$

the rows numbering $1, 2, 3, \ldots\ldots, m-1$, the columns numbering $0, 1, 2, 3, \ldots\ldots, m-1$. In the following examples, this arrangement is adopted, but the letters a, b, c, d with single subscript indices are used for convenience instead of the single letter e with double subscript indices.

EXAMPLES.

$m=3$.

$$\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}(\eta_0)=\begin{pmatrix}a_1 & a_2 & b_1\\ a_2 & b_1 & a_1\end{pmatrix}\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\end{pmatrix}.$$

$$\left.\begin{array}{llll}\text{No. of row,} \quad . \quad . \quad . & (0) & 2 & 1\\ \text{Subscript of } b, \quad . \quad . \quad . & — & 1 & 1\end{array}\right\}$$

$$S=\{(a_1a_2)(b_1)\},$$

$$a_1+a_2+b_1=n,$$

$$b_1+(a_1-b_1)(a_2-a_1)+(a_2-b_1)(b_1-a_2)=0.$$

One linear and one quadratic equation.

[Legendre, "Essai sur la Théorie des Nombres," 2^e^ edition, §480; or, "Théorie des Nombres," 3^e^ edition, §513. Cayley, Collected Mathematical Papers, vol. XI, pp. 86–89, or Proceedings of the London Mathematical Society, vol. XI (1880), pp. 7–9.]

$m = 5.$

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} (\eta_0) = \begin{pmatrix} a_1 & a_4 & b_1 & b_2 & b_1 \\ a_2 & b_1 & a_3 & b_2 & b_2 \\ a_3 & b_2 & b_2 & a_2 & b_1 \\ a_4 & b_1 & b_2 & b_1 & a_1 \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix}.$$

No. of row, . . .	(0)	3	1	4	2
Subscript of b, . . .		1	1	2	2

$$S = \{(a_1 a_2 a_4 a_3)(b_1 b_2)\},$$
$$(a_1 + a_4 + 2b_1 + b_2)\, S^h = n,$$
$$\{b_1 + (a_1 - b_1)(a_2 - a_1) + (a_4 - b_1)(b_1 - a_2) + (b_2 - b_1)(b_2 - a_4)\}\, S^h = 0.$$

Two linear and two quadratic equations.

[Legendre, "Théorie des Nombres," 3e edition, §§523–527. Cayley, Collected Mathematical Papers, vol. XI, pp. 314 and 315, and vol. XII, pp. 72 and 73; or Proceedings of the London Mathematical Society, vol. XII (1881), pp. 15 and 16, and vol. XVI (1885), pp. 61–63. Carey (F. S.), Trinity Fellowship Dissertation, 1884. The present writer has not seen Mr. Carey's Dissertation itself, but the results arrived at by Mr. Carey were quoted by Professor Cayley in his paper of 1885.]

$m = 7.$

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{pmatrix} (\eta_0) = \begin{pmatrix} a_1 & a_6 & b_1 & c_1 & b_3 & c_2 & b_1 \\ a_2 & b_1 & a_5 & c_2 & b_2 & b_2 & c_1 \\ a_3 & c_1 & c_2 & a_4 & b_3 & b_2 & b_3 \\ a_4 & b_3 & b_2 & b_3 & a_3 & c_1 & c_2 \\ a_5 & c_2 & b_3 & b_2 & c_1 & a_2 & b_1 \\ a_6 & b_1 & c_1 & b_3 & c_2 & b_1 & a_1 \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{pmatrix}.$$

No. of row, . . .	(0)	4	1	5	2	6	3
Subscript of b, . . .	—	3	1	2	2	1	3
No. of row, . . ,	(0)	5	3	1	6	4	2
Subscript of c, . . .	—	2	2	1	2	1	1

$$S = \{(a_1 a_3 a_2 a_6 a_4 a_5)(b_1 b_3 b_2)(c_1 c_2)\},$$

$$(a_1 + a_6 + 2b_1 + c_1 + b_3 + c_2)\, S^h = n,$$

$$\begin{aligned}\{b_1 + (a_1 - b_1)(a_2 - a_1) + (a_6 - b_1)(b_1 - a_2) + (c_1 - b_1)(c_2 - a_4)&\\ + (b_3 - b_1)(b_2 - a_5) + (c_2 - b_1)(b_2 - a_6)\}\, S^h &= 0,\end{aligned}$$

$$\begin{aligned}\{c_1 + (a_1 - c_1)\, a_3 + (a_6 - c)\, c_1 + (b_1 - c_1)\, c_2 + (b_3 - c_1)\, b_3 + (c_2 - c_1)\, b_2&\\ + (b_1 - c_1)\, b_3 - (a_2 - c_1)\, a_1 - (b_1 - c_1)\, a_2 - (a_5 - c_1)\, a_3&\\ - (c_2 - c_1)\, a_4 - (b_2 - c_1)\, a_5 - (b_2 - c_1)\, a_6\}\, S^h &= 0.\end{aligned}$$

Three linear and five $(= 3 + 2)$ quadratic equations.

TABLE I.

p	n		η_0	η_1	η_2	η_3	η_4	η_5	η_6
29	4	$\eta_0\eta_1$	1	0	1	0	0	1	1
		$\eta_0\eta_2$	0	1	0	1	1	1	0
		$\eta_0\eta_3$	0	0	1	2	0	1	0
43	6	$\eta_0\eta_1$	0	1	1	1	0	2	1
		$\eta_0\eta_2$	0	1	0	2	1	1	1
		$\eta_0\eta_3$	0	1	2	2	0	1	0
71	10	$\eta_0\eta_1$	2	2	2	0	1	1	2
		$\eta_0\eta_2$	1	2	0	1	3	3	0
		$\eta_0\eta_3$	2	0	1	2	1	3	1

$m = 11.$

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\\ \eta_6\\ \eta_7\\ \eta_8\\ \eta_9\\ \eta_{10}\end{pmatrix}\begin{pmatrix}\eta_0\\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{pmatrix} = \begin{pmatrix} a_1 & a_{10} & b_1 & c_1 & c_4 & c_6 & b_5 & c_5 & c_7 & c_{10} & b_1\\ a_2 & b_1 & a_9 & c_{10} & b_2 & c_3 & c_2 & c_9 & c_8 & b_2 & c_1\\ a_3 & c_1 & c_{10} & a_8 & c_7 & c_8 & b_2 & b_4 & b_3 & c_3 & c_4\\ a_4 & c_4 & b_2 & c_7 & a_7 & c_5 & c_9 & b_4 & b_4 & c_2 & c_6\\ a_5 & c_6 & c_3 & c_8 & c_5 & a_6 & b_5 & c_2 & b_3 & c_9 & b_5\\ a_6 & b_5 & c_2 & b_8 & c_9 & b_5 & a_5 & c_6 & c_3 & c_8 & c_5\\ a_7 & c_5 & c_9 & b_4 & b_4 & c_2 & c_6 & a_4 & c_4 & b_2 & c_7\\ a_8 & c_7 & c_8 & b_3 & b_4 & b_3 & c_3 & c_4 & a_3 & c_1 & c_{10}\\ a_9 & c_{10} & c_2 & c_8 & c_2 & c_9 & c_8 & b_2 & c_1 & a_2 & b_1\\ a_{10} & b_1 & c_1 & c_4 & c_6 & b_5 & c_5 & c_7 & c_{10} & b_1 & a_1 \end{pmatrix}\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\\ \eta_6\\ \eta_7\\ \eta_8\\ \eta_9\\ \eta_{10}\end{pmatrix}$$

No. of row,	(0)	6	1	7	2	8	3	9	4	10	5
Subscript of b,	—	5	1	4	2	3	3	2	4	1	5
No. of row, Subscript of c,	—	4	8	1	5	9	2	6	10	3	7

$$S = \{(a_1 a_2 a_4 a_8 a_5 a_{10} a_9 a_7 a_3 a_6)(b_1 b_2 b_4 b_3 b_5)(c_1 c_2 c_4 c_8 c_5 c_{10} c_9 c_7 c_3 c_6)\}$$

$$(a_1 + a_{10} + 2b_1 + c_1 + c_4 + c_6 + b_5 + c_5 + c_7 + c_{10})\, S^h = n,$$

$$\begin{aligned}\{b_1 + (a_1 - b_1)(a_2 - a_1) &+ (a_{10} - b_1)(b_1 - a_2) + (c_1 - b_1)(c_{10} - a_4) \\ &+ (c_4 - b_1)(b_2 - a_5) + (c_6 - b_1)(c_3 - a_6) + (b_5 - b_1)(c_2 - a_7) \\ &+ (c_5 - b_1)(c_9 - a_8) + (c_7 - b_1)(c_8 - a_9) \\ &+ (c_{10} - b_1)(b_2 - a_{10})\}\, S^h = 0,\end{aligned}$$

$$\begin{aligned}\{c_1 + (a_1 - c_1)\, a_3 &+ (a_{10} - c_1)\, c_1 + (b_1 - c_1)\, c_{10} + \dots \\ &- (a_2 - c_1)\, a_1 - (b_1 - c_1)\, a_2 - (a_9 - c_1)\, a_8 - \dots\}\, S^h = 0.\end{aligned}$$

Five linear and fifteen (= 5 + 10) quadratic equations.

TABLE II.

p	n		η_0	η_1	η_2	η_3	η_4	η_5	η_6	η_7	η_8	η_9	η_{10}
23	2	$\eta_1\eta_0$	1	0	0	0	0	0	0	0	1	0	0
		$\eta_0\eta_2$	0	0	0	0	0	0	1	0	1	0	0
		$\eta_0\eta_3$	0	0	0	0	1	1	0	0	0	0	0
		$\eta_0\eta_4$	0	0	0	1	0	0	0	0	0	1	0
		$\eta_0\eta_5$	0	0	0	1	0	0	0	1	0	0	0
67	6	$\eta_0\eta_1$	1	0	0	2	0	1	1	1	0	0	0
			0	0	0	0	1	0	1	0	1	1	2
			2	2	0	0	0	1	0	1	0	0	0
			0	0	1	0	0	1	0	1	1	1	1
			0	1	0	1	1	0	1	1	0	0	1
89	8	$\eta_0\eta_1$	0	2	1	1	1	0	0	2	0	0	1
			0	1	0	0	1	1	2	0	1	1	1
			2	1	0	0	0	1	1	0	1	1	1
			2	1	1	0	0	2	0	0	0	2	0
			0	0	1	1	2	1	0	2	1	0	0

$m = 13.$

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \\ \eta_7 \\ \eta_8 \\ \eta_9 \\ \eta_{10} \\ \eta_{11} \\ \eta_{12} \end{pmatrix} (\eta_0) = \begin{pmatrix} a_1 & a_{12} & b_1 & c_1 & d_1 & c_4 & c_7 & b_6 & c_6 & c_9 & d_4 & c_{12} & b_1 \\ a_2 & b_1 & a_{11} & c_{12} & b_2 & c_5 & c_2 & d_3 & d_2 & c_{11} & c_8 & b_2 & c_1 \\ a_3 & c_1 & c_{12} & a_{10} & d_4 & c_8 & b_3 & c_{10} & b_5 & c_3 & b_3 & c_5 & d_1 \\ a_4 & d_1 & b_2 & d_4 & a_9 & c_9 & c_{11} & c_3 & b_4 & b_4 & c_{10} & c_2 & c_4 \\ a_5 & c_4 & c_5 & c_8 & c_9 & a_8 & c_6 & d_2 & b_5 & b_4 & b_5 & d_3 & c_7 \\ a_6 & c_7 & c_2 & b_3 & c_{11} & c_6 & a_7 & b_6 & d_3 & c_{10} & c_3 & d_2 & b_6 \\ a_7 & b_6 & d_3 & c_{10} & c_3 & d_2 & b_6 & a_6 & c_7 & c_2 & b_3 & c_{11} & c_6 \\ a_8 & c_6 & d_2 & b_5 & b_4 & b_5 & d_3 & c_7 & a_5 & c_4 & c_5 & c_8 & c_9 \\ a_9 & c_9 & c_{11} & c_3 & b_4 & b_4 & c_{10} & c_2 & c_4 & a_4 & d_1 & b_2 & d_4 \\ a_{10} & d_4 & c_8 & b_3 & c_{10} & b_5 & c_3 & b_3 & c_5 & d_1 & a_3 & c_1 & c_{12} \\ a_{11} & c_{12} & b_2 & c_5 & c_2 & d_3 & d_2 & c_{11} & c_8 & b_2 & c_1 & a_2 & b_1 \\ a_{12} & b_1 & c_1 & d_1 & c_4 & c_7 & b_6 & c_6 & c_9 & d_4 & c_{12} & b_1 & a_1 \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \\ \eta_7 \\ \eta_8 \\ \eta_9 \\ \eta_{10} \\ \eta_{11} \\ \eta_{12} \end{pmatrix}$$

No. of row, . .	(0)	7	1	8	2	9	3	10	4	11	5	12	6
Subscript of b, . . .	—	6	1	5	2	4	3	3	4	2	5	1	6

No. of row, . . Subscript of c, . .	—	9	5	1	10	6	2	11	7	3	12	8	4

No. of row, . .	(0)	10	7	4	1	11	8	5	2	12	9	6	3
Subscript of d, . .	—	4	3	4	1	3	3	2	2	4	1	2	1

$$S = \{(a_1a_2a_4a_8a_3a_6a_{12}a_{11}a_9a_5a_{10}a_7)(b_1b_2b_4b_5b_3b_6)(c_1c_2c_4c_8c_3c_6c_{12}c_{11}c_9c_5c_{10}c_7)(d_1d_2d_4d_3)\},$$

$$(a_1 + a_{12} + 2b_1 + c_1 + d_1 + c_4 + c_7 + b_6 + c_6 + c_9 + d_4 + c_{12})\, S^k = n.$$

$$\{b_1 + (a_1 - b_1)(a_2 - a_1) + (a_{12} - b_1)(b_1 - a_2) + (c_1 - b_1)(c_{12} - a_3) \\ + (d_1 - b_1)(b_2 - a_4) + \ldots.\}\, S^h = 0,$$

$$\{c_1 + (a_1 - c_1)\, a_3 + (a_{12} - c_1)\, c_1 + (b_1 - c_1)\, c_{12} + (d_1 - c_1)\, d_4 + \ldots. \\ - (a_2 - c_1)\, a_1 - (b_1 - c_1)\, a_2 - (a_{11} - c_1)\, a_3 - (c_{12} - c_1)\, a_4 - \ldots.\}\, S^h = 0,$$

$$\{d_1 + (a_1 - d_1)\, a_4 + (a_{12} - d_1)\, d_1 + (b_1 - d_1)\, b_2 + (c_1 - d_1)\, d_4 + \ldots. \\ - (a_3 - d_1)\, a_1 - (c_1 - d_1)\, a_2 - (c_{12} - d_1)\, a_3 - (a_{10} - d_1)\, a_4 - \ldots.\}\, S^h = 0.$$

Six linear and twenty-two (= 6 + 12 + 4) quadratic equations.

TABLE III.

p	n		η_0	η_1	η_2	η_3	η_4	η_5	η_6	η_7	η_8	η_9	η_{10}	η_{11}	η_{12}
53	4	$\eta_0\eta_1$	1	0	0	1	1	1	0	0	0	0	0	0	0
		$\eta_0\eta_2$	0	0	0	0	1	0	0	0	1	0	0	1	1
		$\eta_0\eta_3$	0	1	0	0	0	0	0	1	1	0	0	0	1
		$\eta_0\eta_4$	0	1	1	0	0	0	0	0	0	0	1	0	1
		$\eta_0\eta_5$	0	1	0	0	0	0	0	1	1	0	1	0	0
		$\eta_0\eta_6$	0	0	0	0	0	0	2	0	0	1	0	1	0
79	6	$\eta_0\eta_1$	0	0	1	1	1	1	0	0	1	0	0	0	1
			0	1	0	0	1	1	0	1	0	0	0	1	1
			0	1	0	0	0	0	0	0	1	2	0	1	1
			1	1	1	0	0	0	0	2	0	0	0	0	1
			0	1	1	0	0	0	1	0	1	0	1	1	0
			0	0	0	0	0	1	2	0	1	0	2	0	0
131	10	$\eta_0\eta_1$	0	0	1	2	2	1	0	1	0	2	0	0	1
			2	1	0	0	0	1	0	1	1	0	2	0	2
			0	2	0	0	0	2	1	0	0	1	1	1	2
			0	2	0	0	2	2	0	1	1	1	0	0	1
			1	1	1	2	2	0	0	1	0	1	0	1	0
			2	0	0	1	0	0	2	1	1	0	1	1	1

$m = 17.$

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\\ \eta_6\\ \eta_7\\ \eta_8\\ \eta_9\\ \eta_{10}\\ \eta_{11}\\ \eta_{12}\\ \eta_{13}\\ \eta_{14}\\ \eta_{15}\\ \eta_{16}\end{pmatrix}(\eta_0) = \begin{pmatrix}
a_1 & a_{16} & b_1 & c_1 & d_1 & d_4 & c_6 & d_{11} & c_9 & b_8 & c_8 & d_6 & c_{11} & d_{13} & d_{16} & c_{16} & b_1\\
a_2 & b_1 & a_{15} & c_{16} & b_2 & d_{12} & c_2 & c_5 & d_2 & d_9 & d_8 & d_{15} & c_{12} & c_{15} & d_5 & b_2 & c_1\\
a_3 & c_1 & c_{16} & a_{14} & d_{16} & d_5 & b_3 & c_{10} & d_{14} & c_3 & b_7 & c_{14} & d_3 & c_7 & b_3 & d_{12} & d_1\\
a_4 & d_1 & b_2 & d_{16} & a_{13} & d_{13} & c_{15} & c_7 & b_4 & c_{13} & d_7 & d_{10} & c_4 & b_4 & c_{10} & c_2 & d_4\\
a_5 & d_4 & d_{12} & d_5 & d_{13} & a_{12} & c_{11} & c_{12} & d_3 & c_4 & b_5 & b_6 & b_5 & c_{13} & d_{14} & c_5 & c_6\\
a_6 & c_6 & c_2 & b_3 & c_{15} & c_{11} & a_{11} & d_6 & d_{15} & c_{14} & d_{10} & b_6 & b_6 & d_7 & c_3 & d_2 & d_{11}\\
a_7 & d_{11} & c_5 & c_{10} & c_7 & c_{12} & d_6 & a_{10} & c_8 & d_8 & b_7 & d_7 & b_5 & d_{10} & b_7 & d_9 & c_9\\
a_8 & c_9 & d_2 & d_{14} & b_4 & d_3 & d_{15} & c_8 & a_9 & b_8 & d_9 & c_3 & c_{13} & c_4 & c_{14} & d_8 & b_8\\
a_9 & b_8 & d_9 & c_3 & c_{13} & c_4 & c_{14} & d_8 & b_8 & a_8 & c_9 & d_2 & d_{14} & b_4 & d_3 & d_{15} & c_8\\
a_{10} & c_8 & d_8 & b_7 & d_7 & b_5 & d_{10} & b_7 & d_9 & c_9 & a_7 & d_{11} & c_5 & c_{10} & c_7 & c_{12} & d_6\\
a_{11} & d_6 & d_{15} & c_{14} & d_{10} & b_6 & b_6 & d_7 & c_3 & d_2 & d_{11} & a_6 & c_6 & c_2 & b_3 & c_{15} & c_{11}\\
a_{12} & c_{11} & c_{12} & d_3 & c_4 & b_5 & b_6 & b_5 & c_{13} & d_{14} & c_5 & c_6 & a_5 & d_4 & d_{12} & d_5 & d_{13}\\
a_{13} & d_{13} & c_{15} & c_7 & b_4 & c_{13} & d_7 & d_{10} & c_4 & b_4 & c_{10} & c_2 & d_4 & a_4 & d_1 & b_2 & d_{16}\\
a_{14} & d_{16} & d_5 & b_3 & c_{10} & d_{14} & c_3 & b_7 & c_{14} & d_3 & c_7 & b_3 & d_{12} & d_1 & a_3 & c_1 & c_{16}\\
a_{15} & c_{16} & b_2 & d_{12} & c_2 & c_5 & d_2 & d_9 & d_8 & d_{15} & c_{12} & c_{15} & d_5 & b_2 & c_1 & a_2 & b_1\\
a_{16} & b_1 & c_1 & d_1 & d_4 & c_6 & d_{11} & c_9 & b_8 & c_8 & d_6 & c_{11} & d_{13} & d_{16} & c_{16} & b_1 & a_1
\end{pmatrix}\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\\ \eta_6\\ \eta_7\\ \eta_8\\ \eta_9\\ \eta_{10}\\ \eta_{11}\\ \eta_{12}\\ \eta_{13}\\ \eta_{14}\\ \eta_{15}\\ \eta_{16}\end{pmatrix}$$

No. of row, .	(0)	9	1	10	2	11	3	12	4	13	5	14	6	15	7	16	8
Subscript of b, .	—	8	1	7	2	6	3	5	4	4	5	3	6	2	7	1	8

No. of row, . Subscript of c, .	—	6	12	1	7	13	2	8	14	3	9	15	4	10	16	5	11

No. of row, . Subscript of d, .	—	13	9	5	1	14	10	6	2	15	11	7	3	16	12	8	4

$$S = \{(a_1a_3a_9a_{10}a_{13}a_5a_{15}a_{11}a_{16}a_{14}a_8a_7a_4a_{12}a_2a_6)(b_1b_3b_8b_7b_4b_5b_2b_6)$$
$$(c_1c_3c_9c_{10}c_{13}c_5c_{15}c_{11}c_{16}c_{14}c_8c_7c_4c_{12}c_2c_6)(d_1d_3d_9d_{10}d_{13}d_5d_{15}d_{11}d_{16}d_{14}d_8d_7d_4d_{12}d_2d_6)\}.$$

$$(a_1 + a_{16} + 2b_1 + c_1 + d_1 + d_4 + c_6 + d_{11} + c_9 + b_8 + c_8 + d_6 + c_{11}$$
$$+ d_{13} + d_{16} + c_{16})\, S^h = n,$$

$$\{b_1 + (a_1 - b_1)(a_2 - a_1) + (a_{16} - b_1)(b_1 - a_2)$$
$$+ (c_1 - b_1)(c_{16} - a_3) + (d_1 - b_1)(b_2 - a_4) + \dots\}\, S^h = 0,$$

$$\{c_1 + (a_1 - c_1)\, a_3 + (a_{16} - c_1)\, c_1 + (b_1 - c_1)\, c_{16} + (d_1 - c_1)\, d_{16} + \dots$$
$$- (a_2 - c_1)\, a_1 - (b_1 - c_1)\, a_2 - (a_{15} - c_1)\, a_3 - (c_{16} - c_1)\, a_4 - \dots\}\, S^h = 0,$$

$$\{d_1 + (a_1 - d_1)\, a_4 + (a_{16} - d_1)\, d_1 + (b_1 - d_1)\, b_2 + (c_1 - d_1)\, d_{16} + \dots$$
$$- (a_3 - d_1)\, a_1 - (c_1 - d_1)\, a_2 + (c_{16} - d_1)\, a_3 + (a_{14} - d_1)\, a_4 - \dots\}\, S^h = 0.$$

Eight linear and forty ($= 8 + 16 + 16$) quadratic equations.

TABLE IV.

p	n		η_0	η_1	η_2	η_3	η_4	η_5	η_6	η_7	η_8	η_9	η_{10}	η_{11}	η_{12}	η_{13}	η_{14}	η_{15}	η_{16}
103	6	$\eta_0\eta_1$	1	0	0	1	0	0	0	0	0	1	0	1	0	2	0	0	0
		$\eta_0\eta_2$	0	0	0	0	0	0	1	1	1	1	0	0	1	0	0	0	1
		$\eta_0\eta_3$	0	1	0	0	0	0	1	0	0	1	0	1	0	1	1	0	0
		$\eta_0\eta_4$	0	0	0	0	2	2	0	1	0	0	0	0	0	0	0	1	0
		$\eta_0\eta_5$	0	0	0	0	2	0	0	1	0	0	1	0	1	0	0	1	0
		$\eta_0\eta_6$	0	0	1	1	0	0	0	1	0	1	0	0	0	0	1	1	0
		$\eta_0\eta_7$	0	0	1	0	1	1	1	0	0	0	0	0	1	0	0	1	0
		$\eta_0\eta_8$	0	0	1	0	0	0	0	0	0	1	1	1	0	0	1	0	1
137	8	$\eta_0\eta_1$	0	0	1	1	1	1	0	0	0	0	1	1	1	0	0	0	1
			0	1	0	0	0	0	1	1	0	0	1	0	1	2	0	0	1
			0	1	0	2	0	0	0	1	0	1	1	1	0	0	0	0	1
			0	1	0	0	0	0	2	0	0	2	0	0	0	0	1	1	1
			2	1	0	0	0	0	1	1	0	0	0	0	0	2	0	1	0
			0	0	1	0	2	1	1	1	0	1	0	0	0	0	1	0	0
			0	0	1	1	0	1	1	0	1	1	1	0	0	0	1	0	0
			0	0	0	0	0	0	0	1	2	0	0	1	2	0	1	1	0
239	14	$\eta_0\eta_1$	3	0	2	1	0	1	2	0	0	0	0	0	1	1	1	0	2
			0	2	2	0	1	0	2	1	0	2	1	0	0	0	1	1	1
			0	1	0	0	1	1	1	1	1	0	1	2	2	2	1	0	0
			0	0	1	1	0	1	0	2	1	1	1	1	0	1	1	2	1
			0	1	0	1	1	0	1	0	2	0	1	1	1	1	1	1	2
			0	2	2	1	0	1	2	0	0	2	1	1	1	1	0	0	0
			0	0	1	1	2	0	0	2	0	1	1	1	1	1	1	2	0
			0	0	0	1	1	2	0	0	4	0	2	0	1	0	2	1	0

Memoir on the Substitution-Groups whose Degree does not Exceed Eight.

By G. A. Miller.

Part I.

On the Construction of all the Substitution-groups of a Small Degree.

From the time that Galois demonstrated that the theory of equations is based upon the theory of substitution-groups it has been observed that the determination of all the possible substitution-groups of a given degree is a problem of fundamental importance in algebra.[*] Although this problem has been studied by a large number of mathematicians, yet little progress has been made towards its complete solution. It is well known that all the groups of a given degree may be found by tentative methods, but for large degrees the number of trials that may be necessary becomes extremely large.

During recent years a large number of substitution-groups, which were usually found by tentative processes, have been published. All the possible groups whose degree is less than eleven have been determined, and the transitive groups, especially the primitive ones, have been determined for considerably larger degrees.[†] In what follows we shall aim to give enough of the general theory of group construction to find all the possible groups whose degree does not exceed eight without any tentative processes.

In the earliest work that devotes considerable attention to substitution-groups, *Teoria generale delle equazioni, in cui si dimostra impossibile la soluzioni algebraica delle equazioni generali di grado superiore al quarto*, di Paolo Ruffini,[‡]

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[*] Cf. Serret, "Algèbre supérieure" (1866), p. 256.

[†] Jordan, Comptes rendus, vol. LXXV, 1872, p. 175–7. Miller, American Journal, vol. XX, 1898, p. 229.

[‡] A review of the group theory that is contained in this work is given by Burkhardt, Schlömlich Zeitschrift, 1892, supplement, p. 159.

Bologna, 1799, we find that these groups are divided into two classes, viz. the transitive and the intransitive groups. The former of these classes is subdivided into the primitive and the imprimitive groups. These divisions have been adopted by all subsequent writers, and they will be employed in what follows.

When one of two substitution-groups can be transformed into the other, each of the two groups is called the transform of the other. In the enumeration of the possible substitution-groups, all of these transform groups are considered to be identical. Two substitution-groups are distinct or different when it is impossible to transform the one into the other.* Two or more different substitution-groups may represent the same abstract group.

It may happen that the substitutions of a group are completely determined by those of a subgroup. E. g. the substitutions of the non-cyclical regular group of order 6 are completely determined by those of its subgroup of order 3, while those of the regular cyclical group of this order are not fully determined by this subgroup, since there are six substitutions of degree and order 6 whose squares are contained in the given subgroup of order 3.

If the substitutions of a transitive group, which is known as an abstract group and whose order is composite, are completely determined by those of each one of its subgroups besides identity, the group must be regular and contain no substitution whose order is less than the order of the group divided by 2. Hence there are only two such groups, viz. the two regular groups of order 4. It is evident that the substitutions of an intransitive group are never completely determined by those of each one of its subgroups.

As an illustration of the importance of knowing whether all the substitutions of a group are completely determined by those of a given one of its subgroups, we may prove that there are no more than one group of degree eight and order 1344. Such a group must contain the doubly transitive simple group of degree seven and order $1344 \div 8 = 168$, and hence it must be triply transitive. It must, therefore, contain an intransitive subgroup of order 48 whose transitive constituents are of degrees two, six and whose transitive subgroup of order 24 and degree six is positive.† The substitutions of this subgroup of order 24 determine all the substitutions of the transitive group of order 48 and degree six, since they do not form a self-conjugate subgroup of a larger group of this degree;[*] each of these groups of order 1344 must, therefore, contain substitutions of degree eight which are fully determined by the substitutions of the given subgroup of degree seven and hence there cannot be more than one such group.

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* Cf. Burnside, Messenger of Mathematics, vol. XXVIII (1898), p. 103.

† Miller, Bulletin of the American Mathematical Society, vol. IV, 1898, p. 140.

§1.—*Intransitive Groups.*

An intransitive group contains two or more transitive constituents. Since the degree of each of these constituents is less than the degree of the group, we may suppose that the transitive constituent groups of the required intransitive groups are known. The first problem which we shall consider may be stated as follows: To determine all the possible substitution-groups that may be formed by combining intransitively a given system of transitive groups, two groups being considered as identical whenever one can be transformed into the other by means of some substitution.[†]

If an intransitive group contains more than two transitive constituents, any number of these constituents form a group which has an $1, \alpha$ isomorphism to the entire group and an α_1, α_2 isomorphism to the group formed by the remaining constituent or constituents. All such intransitive groups may therefore be constructed by establishing isomorphisms between the intransitive groups that contain all except one of the constituents of the required groups and the transitive group which is the remaining constituent For instance, to find all the possible intransitive groups of degree ten whose constituents are of degrees four, three, three, we have only to consider the isomorphisms between the intransitive groups of degree seven that involve transitive constituents of degrees four, three and the transitive groups of degree three. We can also obtain all these groups by considering all the possible isomorphisms between the intransitive groups of degree six that involve two transitive constituents of degree three and the transitive groups of degree four.

The form in which a group can be represented as an intransitive substitution-group depends upon its abstract group properties. It must contain at least one transitive constituent whose order is equal to the order of the entire group unless it contains two or more self-conjugate subgroups (differing from identity) which have only identity in common. If it contains two such self-conjugate subgroups, it can always be represented as an intransitive group which involves no transitive constituent whose order is equal to the order of the group. As such transitive constituents we may use the quotient groups (represented as substitution-groups) with respect to any two of the given self-conjugate subgroups. Since these remarks apply to the constituent groups as well as to the entire intransitive group, it follows that the form in which a group is represented as an intransitive substitution-group frequently throws light on the properties of the group.

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[*] When a self-conjugate subgroup of a group of degree n is not contained in any larger group of this degree as a self-conjugate subgroup, its substitutions must determine all the substitutions of the group, but the converse is not always true.

[†] Cf. Bolza, American Journal of Mathematics, vol. XI, 1889, pp. 195–214.

Let $G_{g_1}^{m_1}$ and $G_{g_2}^{m_2}$ represent two substitution-groups of degrees m_1 and m_2, and of orders g_1 and g_2 respectively, and suppose that none of the m_1 letters in the first of these two groups is found in the second. If we multiply each of the substitutions of one of these groups by all of the substitutions of the other, we obtain an intransitive group of degree $m_1 + m_2 = n$ and of order $g_1 g_2 = g$. This group G_g^n is said to be the *direct product*[*] of $G_{g_1}^{m_1}$ and $G_{g_2}^{m_2}$. It is evident that any abstract group which is the direct product of two groups may be represented as an intransitive substitution-group in which these subgroups are transitive and do not contain a common element.

The groups $G_{g_1}^{m_1}$ and $G_{g_2}^{m_2}$ are self-conjugate subgroups of G_g^n and they contain no common substitution besides identity. If both of these subgroups are simple and if $g_1 \neq g_2$ or if at least one of the subgroups is non-Abelian, G_g^n does not contain any other self-conjugate subgroup besides identity. In general, if k_1 and k_2 represent the numbers of the self-conjugate subgroups of $G_{g_1}^{m_1}$ and $G_{g_2}^{m_2}$ respectively, then will G_g^n contain just $(k_1 + 1)(k_2 + 1) - 1$ self-conjugate subgroups that are the direct products of a group contained in each of the given constituents. In this enumeration identity is considered a self-conjugate subgroup, but the entire group is not included in this term. Other self-conjugate subgroups of G_g^n can exist only when a quotient group of one of the given transitive constituents, or one of its subgroups, is simply isomorphic to a quotient group of the other or one of its subgroups, and when all the operators of each constituent are commutative to each of the operators of this quotient group. It is evident that these conditions are sufficient as well as necessary. Somewhat similar remarks apply to all the possible subgroups of G_g^n.[†] We shall incidentally consider some of these subgroups in what follows.

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[*] Cf. Hölder, Mathematische Annalen, vol. XLIII, p. 330.

[†] Cf. Klein-Fricke, Modulfunktionen, vol. I, 1890, p. 403.

Since each of the substitutions of one of the two subgroups $G^{m_1}_{g_1}$, $G^{m_2}_{g_2}$ is commutative to every substitution of the other, the order of any substitution of G^n_g is the lowest common multiple of the orders of its constituents that belong to these subgroups. In particular, it follows that the necessary and sufficient condition that G^n_g is cyclical is that g_1, g_2 are prime to each other, and that the given subgroups of these orders are cyclical. G^n_g is evidently Abelian when the two given subgroups have this property. These remarks clearly apply also to the case when $G^{m_1}_{g_1}$ and $G^{m_2}_{g_2}$ represent the same group written in different sets of letters. In this case G^n_g is said to be the square of one of these subgroups.

The intransitive group of order g_1g_2 which we have just considered can always be constructed regardless of the particular groups which $G^{m_1}_{g_1}$ and $G^{m_2}_{g_2}$ may represent. Other intransitive groups having these constituents can be constructed only when certain conditions are satisfied, and when they are possible they must be subgroups of G^n_g. We proceed to consider the necessary conditions.

Suppose that G^n_m is any intransitive group with the constituents $G^{m_1}_{g_1}$ and $G^{m_2}_{g_2}$. All the substitutions of $G^{m_1}_{g_1}$ which correspond to a self-conjugate subgroup of $G^{m_2}_{g_2}$ must form a self-conjugate subgroup of $G^{m_1}_{g_1}$. In particular, the substitutions of one of the transitive constituents which correspond to identity in the other transitive constituent must form a self-conjugate subgroup. It is also evident that the number of the substitutions in one of these constituents which correspond to identity in the other constituent is equal to the number that correspond to any other substitution in the second constituent. Hence we see that if H_1 is the self-conjugate subgroup of $G^{m_1}_{g_1}$ that corresponds to identity in $G^{m_2}_{g_2}$ and if H_2 is the self-conjugate subgroup of $G^{m_2}_{g_2}$ that corresponds to identity in $G^{m_1}_{g_1}$, then must the quotient group of $G^{m_1}_{g_1}$ with respect to H_1 be simply isomorphic to the quotient group of $G^{m_2}_{g_2}$ with respect to H_2 and m must equal $h_1g_2 = h_2g_1$, h_1 and h_2 being the orders of H_1 and H_2 respectively.

The construction of the intransitive groups which contain two given constituents is thus reduced to two cases, viz.: 1), forming the direct product of the substitutions of the constituents, and 2), forming all the possible simple isomorphisms between the constituent groups and their quotient groups, which lead to substitution-groups that cannot be transformed into each other. The former of these cases is very simple and requires no further attention, while the latter is generally much more complex whenever such groups are possible.

We have observed that a necessary condition to construct an intransitive group besides G^n_g with the given constituents is that their quotient groups with respect to H_1 and H_2 (where H_1 or H_2 or both of them may be identity) are the same abstract group. This condition is evidently sufficient as well as necessary for the construction of G^n_m, where m is a factor of g and less than g. When the order of the given quotient group exceeds 2, it has more than one simple isomorphism to itself. In this case it is necessary to consider the intransitive groups which correspond to all the possible simple isomorphisms of the given quotient group to itself. Since all the cogredient isomorphisms lead to the same group, the total number of these intransitive substitution-groups cannot exceed the quotient of the total number of the simple isomorphisms of the quotient group to itself divided by the number of the cogredient ones.

If the given quotient group is cyclical, there is clearly only one group with respect to the given self-conjugate subgroups. If this quotient group is complete,* it admits only cogredient simple isomorphisms to itself, and hence there cannot be more than one group for H_1 and H_2. When either H_1 or H_2 is identity and the corresponding constituent group is regular, there cannot be more than one group, since every simple isomorphism of a regular group to itself can be obtained by transforming the group by certain operators.†

In general, we may obtain all the possible intransitive groups with respect to the self-conjugate subgroups H_1 and H_2 in the following manner: First determine the group of isomorphisms of the given quotient group and find the operators of this group of isomorphisms that correspond to all the possible conjugates of a given intransitive group. If these do not include all the operators of the group of isomorphisms, form an isomorphism that corresponds to another operator of this group. We thus obtain a second intransitive group, and all the conjugates of this group correspond to operators of the group of isomorphisms which are distinct from those that correspond to the first intransitive group. If this does not exhaust all the operators of the group of isomorphisms of the given quotient group, we may form a third intransitive group, etc. We continue this process until all the operators of the group of isomorphisms are exhausted.

When the two transitive constituents are conjugate, the set of operators of the group of isomorphisms which corresponds to the conjugates of any intransitive group must include all the inverse operators of the set, since the inverse of

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* Hölder, Mathematische Annalen, vol. XLVI, 1895, p. 325.

† Cf. Netto, "Theory of Substitutions," 1892, p. 110.

DEGREE.	ORDERS.				
	2	4	8	16	32
4	1	1			
6	1	2	1		
8	1	3	3	1	
10	1	4	6	4	1

This rule does not apply to the groups of a higher degree, since there are six such groups of degree twelve and order 4.*

§2.—*Imprimitive Groups.*

The substitutions of an imprimitive group must permute all its possible systems of imprimitivity according to a transitive substitution group. Hence each system must contain the same number of elements; i. e., it is impossible to construct an imprimitive group of a prime degree. If the transitive group, according to which the systems of imprimitivity are permuted, is itself imprimitive, it is possible to combine these systems until we obtain systems which are permuted according to a primitive group.† Hence every imprimitive group must have an α, 1 isomorphism to some primitive group which corresponds to the permutations of one set of its systems of imprimitivity when its substitutions are transformed by every substitution of the group.

The subgroup of order α which corresponds to identity in the given primitive group must be self-conjugate. If its order exceeds unity, it must be intransitive, and its systems of intransitivity are also systems of imprimitivity of the given group. It should, however, be observed that the latter systems of imprimitivity are not necessarily permuted, by all the substitutions of the group, according to the given isomorphic primitive group. In case the systems are permuted according to an imprimitive group, they may always be united into larger systems which are permuted according to the given isomorphic primitive group.[*] If, however, this primitive group is regular, the systems of intransitivity of the given subgroup of order α must coincide with the former systems of imprimitivity, since all the systems of intransitivity of the given subgroup must be permuted transitively by all the substitutions of the given imprimitive group.

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* Philosophical Magazine, vol. XLI, 1896, p. 435.

† Jordan, "Traité des Substitutions," 1870, p. 399.

Every group of a finite order may be represented as a regular substitution group, and every regular group of composite order is imprimitive.[†] Hence every group of a finite order, with the exception of the trivial case where the order is a prime number, may be represented as an imprimitive substitution-group whose degree is equal to its order. This representation is unique; i. e., there is one and only one regular substitution-group which is simply isomorphic to any given group of a finite order. Some groups can also be represented as non-regular, imprimitive groups. The properties of these groups will be considered later.

It is convenient to have a special name for the subgroup of order α which contains all the substitutions of the group that do not permute any of the systems of imprimitivity. It has been called the *base* or the *head* of the group. We shall generally use the latter term. All the substitutions of an imprimitive group that are not contained in the head are said to form the *tail* of the group. We have already observed that each of the transitive constituents of the head must be of the same degree. Since all of them are transformed transitively by the substitutions of the imprimitive group, they must be similar to each other.

It is well known that every non-Abelian regular group is conjugate to a group containing the same letters whose substitutions are commutative to every substitution of the regular group. Each substitution of one of these groups determines systems of imprimitivity of the other group. The systems of imprimitivity obtained in this way do not necessarily include all the possible systems. When a regular group of composite order is Abelian, its own substitutions determine different systems of imprimitivity. Hence it follows that the elements of a group can sometimes be arranged into systems of imprimitivity in a large number of ways.

1.—*The Head of an Imprimitive Group.*

It is sometimes impossible to find any systems of imprimitivity of a given imprimitive group, which are not permuted among themselves by every substitution of the group, that differs from unity. Such imprimitive groups occur for degrees twelve and fourteen and for higher degrees, but not for any lower degrees. In these cases we say that the only possible head of the group is identity. In what follows we shall consider the heads whose orders exceed unity. We have already observed that the transitive constituents of these heads are similar groups, i. e. the heads are obtained by establishing some isomorphism between a transitive group written in two or more different sets of elements. It remains only to inquire into the nature of this isomorphism.

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[*] Cf. Miller, Quarterly Journal of Mathematics, vol. XXIX, p. 235.

[†] Jordan, "Traité des Substitutions," 1870, p. 60.

The simplest possible cases are those in which we merely form the product of the groups obtained by writing a given transitive group in different sets of letters or establish a simple isomorphism between the corresponding substitutions of these groups. These two heads are always possible, regardless of the number of systems of imprimitivity. Other heads containing a given transitive constituent can be constructed only when this transitive constituent satisfies certain conditions.

If a transitive constituent of a head is compound, it is easy to form other heads by merely multiplying the divisions with respect to the same self-conjugate subgroup in the different sets of letters. The heads thus obtained are all symmetrical with respect to the transitive constituents. When the primitive group, according to which the systems of imprimitivity are permuted, is not symmetrical, it is simply necessary that the head allows the permutations indicated by this primitive group. It is thus frequently possible to construct imprimitive groups whose heads do not permit the symmetric permutation of its systems of intransitivity.*

2.—*Groups containing Two Systems of Imprimitivity.*

One-half of the substitutions of such a group must transform each of these systems into itself. These must form an intransitive group which differs from unity, since the degree of an imprimitive group cannot be less than four. Let G_g^n be a transitive constituent of the head. We shall first suppose that $g = n!$ and that the head is the product of its constituents. By adding to this head a substitution of order 2 that merely interchanges the corresponding elements of its constituents we obtain an imprimitive group of order $2(n!)^2$ This group evidently includes all the imprimitive groups of degree $2n$ that contain two systems of imprimitivity, and it includes no other transitive group of this degree. Hence each of the substitutions that is contained in the tail of one of these imprimitive groups is of degree $2n$, and it involves only negative cycles. It must clearly satisfy the following three conditions: (α), interchange the systems of intransitivity of the head; (β), transform the head into itself; (γ), its square must be contained in the head. These conditions are sufficient as well as necessary. The factors of composition of these imprimitive groups are clearly obtained by adding 2 to the factors of composition of the head.

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* Cf. Miller, Quarterly Journal of Mathematics, 1896, vol. XXVIII, p. 197.

We proceed to establish theorems which are useful in constructing the imprimitive groups that involve only two systems of imprimitivity. Several of these theorems are included in the more general theorems of the following sections, and are given here for the purpose of leading up to them. We shall represent the n elements of the two systems by $a_1, a_2, a_3, \ldots, a_n$ and $a'_1, a'_2, a'_3, \ldots, a'_n$ respectively, and we shall use s_a, s'_a to represent the same substitution in the two systems, the elements of s_a belonging to the first system and those of s'_a to the second. Similarly we shall employ G, G' respectively to represent the same group in the two systems. The substitution of order 2 which merely permutes the corresponding elements of the two systems we shall represent by t. Hence we have $t = a_1a'_1 . a_2a'_2 . a_3a'_3 \ldots a_na'_n$, $t^{-1}s_at = s'_a$, $t^{-1}Gt = G'$.

Theorem I. *If one of two transitive constituents of the head of an imprimitive group is G, and if H is the largest group involving the same elements as G that transforms G into itself, this imprimitive group is included in the imprimitive group generated by H, H', t.*

We may choose the two transitive constituents of the head in such a way that t transforms one into the other. Since the remaining generator of the imprimitive group in question must also have this property, the factor which must be multiplied into t to obtain this generator, must transform each of the given constituents of the head into itself. Hence this factor must be contained in the product of H and H'.

Theorem II. *The squares of the substitutions of the tail of the smaller imprimitive group of the preceding theorems are transforms, with the respect to substitutions of H, of the substitutions obtained by making H simply isomorphic to H' in such a way thas t is commutative to every substitution of this group. When H coincides with G, all these transforms are squares of substitutions in the tail of the group generated by H, H', t.*

We may represent any substitution of the tail of the given group by $s_\alpha s'_\beta t$, where s_α, s'_β belong to H, H' respectively according to the preceding theorem. Hence

$$(s_\alpha s'_\beta t)^2 = s_\alpha s_\beta s'_\beta s'_\alpha = s_\beta^{-1} s_\beta s_\alpha s'_\beta s'_\alpha s_\beta .$$

Since $s_\beta s_\alpha s'_\beta s'_\alpha$ is in the given simple isomorphisms of H to H' and s_β belongs to H, the first part of the theorem is proved. It remains to show that the squares of the substitutions in the tail of H, H', t give all the transforms with respect to substitutions of H of the substitutions of the group obtained by making H simply isomorphic to H' in such a way that identical substitutions correspond whenever H and G are identical. In other words, that it is possible to choose s_α, s_β in such a manner as to satisfy the equation $s_\alpha s_\beta s'_\beta s'_\alpha = s_\gamma^{-1} s_\delta s'_\delta s_\gamma$, where s_δ and s_γ are any given substitutions of H. If we let $s_\alpha = s_\gamma^{-1} s_\delta$ and $s_\beta = s_\gamma$, this equation is satisfied and the proof is complete.

Theorem III. *If G is any transitive group, there is always one and only one imprimitive group whose head is the direct product of G and G'.*

Let $s_\alpha s'_\beta t$ be any substitution that may be added to this head to generate an imprimitive group. Since the square of this substitution must be found in the head, we have $(s_\alpha s'_\beta t)^2 = s_\alpha s_\beta s'_\beta s'_\alpha =$ some substitution of the head. Hence either, both or neither of the substitutions s_α, s_β belong to G. In the former case we evidently have an imprimitive group; in the latter case it is necessary that s_α, s_β correspond to the inverse operators of the quotient group of H with respect to G, H being the largest group (involving the same elements as G) that transforms G into itself. Hence $s_\beta = s_\alpha^{-1} s$ where s is some substitution of G.

If we transform t by the inverse of s_α we obtain $s_\alpha t s_\alpha^{-1} = s_\alpha t s_\alpha^{-1} t t = s_\alpha s'^{-1}_\alpha t$. Since s_α transforms the given head into itself, we observe that the groups obtained in the second case of the preceding paragraph are conjugate to that of the first case. From this it follows that all the groups imprimitive that contain the given head are conjugate.

Theorem IV. *If G has a self-conjugate subgroup which is also a self-conjugate subgroup of H, and if no two of the m divisions of H with respect to this self-conjugate subgroup are similar, then there are $g \div k = l$ imprimitive groups having for a common head the group obtained by the simplest k, k isomorphism between G and G', where g and k are the order of G and the given self-conjugate subgroup respectively.*

Let $s_1, s_2, \ldots, s_n$ satisfy the condition that one of them belongs to each one of the given divisions of H. Each one of the substitutions $s_1 t$, $s_2 t$, $s_3 t$, $\ldots$, $s_m t$

transforms the given head into itself, but only l of them (say the first l) have their squares in this head. The l substitutions combined separately to the head lead to l imprimitive groups. These groups must be distinct because the squares of the substitutions of the tails are not similar. The remaining substitutions of the product of H and H' may be obtained by multiplying the given head into each of the following substitutions:

$$s_\beta s'_\beta s_\alpha t, \quad \alpha = 1, 2, \ldots, m, \quad \beta = l + 1, 2l + 1, \ldots, m - l + 1.$$

From
$$s_\beta^{-1} s_\beta s'_\beta t s_\beta = s'^2_\beta s_\alpha t$$
it follows that these are conjugate with respect to substitutions that transform the given head into itself, to the substitutions considered above. Hence l is the total number of the imprimitive groups containing the given head.

Theorem V. *There is only one imprimitive group whose head is obtained by making the symmetric group whose degree is not two or six simply isomorphic to itself. If the degree of the symmetric group is two or six, there are two such groups.*

Since the symmetric group whose degree is not two or six, is a complete group,* we can form only one intransitive group by making such a symmetric group simply isomorphic to itself. We may suppose that in this intransitive group the corresponding substitutions are identical in the two systems. If we add to this group a substitution of order 2 which simply interchanges the corresponding elements of the two systems (t), we obtain an imprimitive group whose order is twice the order of the given symmetric group.

Suppose that a different tail to the given head was generated by the head and the substitution s. This substitution must interchange the two systems of intransitivity of the head. Hence we have

$$st = s_1 \text{ or } s = s_1 t,$$

where s_1 must transform the given head into itself without permuting its systems of intransitivity, hence the second tail would contain a substitution of the form $s_2 t$, where s_2 involves only elements from one of the systems of intransitivity of the head. Since the head is a complete group, this is impossible, unless $s_2 \equiv 1$. In this case we have the group given above. As an abstract group this is the direct product of the given symmetric group, and an operator of order 2.

When the degree of the symmetric group is six, we can construct two intransitive groups by making it simply isomorphic to itself. To each one of the heads obtained in this way, we can add only one tail for the same reason as was given above. Hence there are just two such groups.* When the degree is two, we evidently obtain the two regular groups of order 4.

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* Hölder, Mathematische Annalen, 1895, vol. XLVI, pp. 345 and 325.

Theorem VI. *There are only two imprimitive groups whose common head is the group obtained by making the alternating group whose degree is not six simply isomorphic to itself. If the degree of the alternating group is six, there are four such groups.*

When the degree of the alternating group differs from six, we can obtain only one intransitive group by making it simply isomorphic to itself.† We may, therefore, suppose that t is commutative to every substitution of the given head. It is evident that t and the given head generate an imprimitive group which is simply isomorphic to the direct product of the given alternating group and an operator of order 2.

As under the preceding theorem, we may suppose that s and this head generate another imprimitive group, and we find by the same process of reasoning that s_1 must transform the given head into itself without interchanging its systems of intransitivity. We can, however, not assume that s_2 involves only elements from one of the systems; but, since the symmetric group of any degree larger than three contains no substitution besides identity that is commutative to each one of the substitutions in its alternating subgroup, we must assume that t is commutative to s_2. If s_2 is not found in the given head, all its possible values may be found by multiplying this head by s_2. Hence there are no more than two such groups.

That the two possible imprimitive groups just found are really distinct, follows directly from the fact that the latter is simply isomorphic to the symmetric group while the former does not have this property. Some of these arguments do not apply to the case when the degree of the alternating group is three, but in this case we obtain the well-known two regular groups of degree six. When the degree of the given alternating group is six, it is possible to construct two intransitive groups by making it simply isomorphic to itself. To each one of these heads we may add two tails and thus obtain the given four

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[*] Miller, Bulletin of the American Math. Society, 1895, vol. I, p. 258.

[†] Hölder, Mathematische Annalen, 1895, vol. XLVI, p. 340; cf. Miller, loc. cit.

imprimitive groups. These groups are explained in connection with the groups whose head is the symmetric group to which references are given above.

From the preceding theorems it follows that if one of the two transitive constituents of the head is the symmetric group of degree n, where n equals 3, 5 or exceeds 6, there are only 4 possible imprimitive groups. According to theorems III and V, there is only one such group for each of the two heads obtained by multiplying the two transitive constituents of the head and by making them simply isomorphic. According to theorem IV, there are two such groups whose head consists of the positive substitutions in the product of its constituents. When n is greater than 6 or $n = 3, 5$, we can evidently form no other head with the given constituents, and hence no additional imprimitive group of this type.

When $n = 2$, there are evidently only two possible heads. When the head is of order two, we may obtain the two regular groups of order 4, and when it is of order 4, there is only one group according to theorem III. Hence there are only three imprimitive groups of degree four. When $n = 4$, the given constituent group is isomorphic to the symmetric group of degree three. We thus obtain one more head than occurs in the general case. It follows directly from theorem V that we can add only one tail to this head. Hence there are five imprimitive groups of degree eight in which a transitive constituent of the head is the symmetric group of degree four. When $n = 6$, there is one head that does not occur in the general case. This case was considered under theorem V. Hence there are five imprimitive groups of degree twelve in which one of the transitive constituents of the head is the symmetric group of degree six. This completes the study of the imprimitive groups of two systems whose heads contain the symmetric group as a transitive constituent.

We shall now consider all the possible imprimitive groups of degree $2n$ whose heads contain the alternating group of degree n as a transitive constituent. For $n = 2$ there evidently is no such group. When $n = 3$, there are the two regular groups of order 6 and the group of order 18 whose head is the product of two alternating groups of degree three. When $n = 4$, we may construct three heads, since the alternating group of degree four is isomorphic to the alternating group of degree three. The groups which contain either one of two of these heads have been considered under the preceding theorems. It remains to consider what tails may be added to the head of order 48.

If we represent the substitution of order two which merely interchanges the corresponding elements of the two systems by t, we observe that t and this head generate an imprimitive group of order 96. All the other possible tails can be generated by st, where s transforms the head into itself without permuting its systems of intransitivity and $(st)^2$ is contained in the head. Since we might use the product of any substitution in the head into st, in place of st we may suppose that s either belongs to the first system of the head or that it is a particular substitution of the second system. Hence we can add only two tails that are not conjugate to this head, i. e. there are five imprimitive groups of degree eight whose heads contain the alternating group of degree four as a transitive constituent.

When $n = 6$, we have seen that there are also five imprimitive groups of the type in question. For all the values of n which differ from those that have just been considered, we can construct only the two heads which are included in theorems III and VI. Hence *there are, in general, only three imprimitive groups of degree* $2n$ *whose heads contain the alternating group of degree* n *as a transitive constituent;* when $n = 4$ or 6, *there are five such groups.*

All of the preceding groups are evidently also distinct as abstract groups, and when n exceeds 4, one of the factors of composition must be $n! \div 2$, hence all of these groups are insolvable whenever their degree exceeds eight. When their degree does not exceed eight, they are evidently solvable.

3.—*Groups containing more than Two Systems of Imprimitivity.*

We have seen that every imprimitive group must contain systems of imprimitivity which its substitutions transform according to an isomorphic primitive group (P). The imprimitive group is either simply or multiply isomorphic to P. In the former case it contains no substitution besides identity that transforms each of these systems of imprimitivity into itself, while in the latter case it must contain an intransitive self-conjugate subgroup which is composed of all the substitutions of the group that transform each of the given systems of imprimitivity into itself and which corresponds to identity of P in the given isomorphisms. If the systems of intransitivity of this self-conjugate subgroup differ from the given systems of imprimitivity, they may be united as units in such a manner as to form these systems.

If the required imprimitive group (G) is simply isomorphic to P, the latter (which is supposed to be known) must contain non-maximal subgroups that include no self-conjugate subgroup of P besides identity, and whose order is the quotient obtained by dividing the order of P by the degree of G. The required imprimitive groups can therefore not be constructed unless P contains at least one system of conjugate subgroups that satisfy the given condition, and in this case the number of the possible imprimitive groups is known.*

Every self-conjugate subgroup of P must be transitive, since a primitive group cannot contain an intransitive self-conjugate subgroup. If n and α represent respectively the degree of P and the number of elements in each of the systems of imprimitivity of G, and if P contains regular subgroups, each of the corresponding subgroups of G will contain α systems of intransitivity. If one of these regular subgroups of G is self-conjugate, G must also contain α systems of imprimitivity, viz. the systems of intransitivity of the subgroup that corresponds to the given regular subgroup of P. For instance, each of the imprimitive groups of degrees six and eight that are simply isomorphic to the symmetric groups of degrees three and four respectively must also contain two systems of imprimitivity.

When P is contained in the metacyclic group of degree p, G must contain α systems of imprimitivity as well as p systems, and when P is the symmetric group of degree $n > 2$, there is always one and only one imprimitive group of degree $2n$ that contains n systems of imprimitivity, and is simply isomorphic to P. This group contains also two systems of imprimitivity. The total number of the imprimitive groups that are simply isomorphic to the symmetric group of degree n when n is not equal to 6, is clearly equal to the total number of the substitution-groups whose degree does not exceed $n-1$, excluding identity, increased by the number of the substitution-groups of degree n that are not maximal subgroups of the symmetric group of this degree.† We proceed to establish some general theorems with respect to the imprimitive groups containing more than two systems of imprimitivity and which are not simply isomorphic to the primitive group according to which these systems are permuted.

Theorem I. *If each of the systems of imprimitivity includes a prime number*

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* Jordan, "Traité des substitutions" (1870), p. 57; cf. Miller, Bulletin of the American Mathematical Society, vol. III (1897), p. 215.

† Dyck, Mathematische Annalen, vol. XXII (1882), p. 91; cf. Proceedings of the American Philosophical Society, vol. XXXVI, 1897, p. 208.

of elements, the elements of the transitive constituents of the head must be the systems of imprimitivity of the group.

Since these transitive constituents must also be systems of imprimitivity, each of them must be composed of the same number of elements. If they would not be the given systems of imprimitivity, these could be obtained by combining the degrees of some of the transitive constituents taken as units. This is impossible, since the given systems of imprimitivity are supposed to be prime.

Theorem II. *An imprimitive group of degree n whose systems of imprimitivity are transformed by all the substitutions of the group according to a primitive group and whose head is not identity, must transform the transitive constituents of this head either according to the given primitive group or according to an imprimitive group that is simply isomorphic to this primitive group and whose systems of imprimitivity are transformed according to this primitive group.*

The systems of intransitivity of the head must be transformed according to a transitive group whose order cannot differ from the order of the primitive group according to which the systems of imprimitivity of the group are transformed, since these systems of imprimitivity could not be permuted without permuting the systems of intransitivity of the head. By means of these theorems the construction of all the possible imprimitive groups of degree n whose head is not identity, is reduced to the construction of the groups that permute the systems of intransitivity of given self-conjugate intransitive subgroups according to known groups. E. g. to construct all such imprimitive groups of degree twelve it is only necessary to construct the groups that contain a self-conjugate subgroup involving two systems of intransitivity, those that contain self-conjugate subgroups involving three systems of intransitivity, those that contain self-conjugate subgroups involving four systems of intransitivity which are transformed according to the symmetric or the alternating group of degree four, and those involving six systems of intransitivity which are transformed according to a primitive group of degree six or according to the non-Abelian regular group of this degree. In the last case the group would also contain two systems of imprimitivity, and hence this does not need to be considered for this particular degree.

Theorem III. *If one of the m transitive constituents of the head of an imprimitive group is G, and if H is the largest group involving the same elements as G that transforms G into itself, this imprimitive group is included in the imprimitive group whose head is the direct product of mH's written in different systems of elements and* *whose tail permutes these H's in the same way as the given imprimitive group permutes the given G's.*

If t is any substitution in the tail of the required imprimitive group, it must possess the following properties: 1) transform the given head into itself, 2) permute its systems of intransitivity according to a given substitution, and 3) its first power that leaves all these systems of intransitivity unchanged must be contained in the head. Since the group whose head is the direct product of the mH's includes all the substitutions that satisfy the first and the second of these three conditions, it must include the required group.

Corollary. *When the head of an imprimitive group is the direct product of transitive groups which are not self-conjugate in any larger group of the same degree, there is only one imprimitive group that permutes the transitive constituents of the head according to a given group.*

Theorem IV. *There is only one imprimitive group whose head is the direct product of a system of m conjugate transitive groups written in different sets of elements and whose tail permutes all these transitive groups cyclically.*[*]

According to the preceding theorem, such a group must be contained in the imprimitive group whose head is the direct product of mH's and whose tail is generated by a substitution (t) of order m which permutes the corresponding elements of the H's in order. Hence its tail is generated by

$$s_1^1 s_2^2 \ldots . s_m^m t,$$

where the upper index indicates the system and the lower the particular substitution of the system. Since the m^{th} power of this substitution is contained in the head of the required group, we have

$$s_1^1 s_2^1 s_3^1 \ldots . s_m^1 = h,$$

where h is some substitution in the first transitive constituent of the head. If this condition is satisfied, the group generated by the given head and $s_1^1 s_2^2 s_3^3 \ldots . s_m^m t$ is conjugate to the group generated by this head and t; i. e. there is only one such imprimitive group.

§3.—*Primitive Groups.*

Every transitive group of degree n contains $n \div \alpha$ conjugate subgroups of degree $n - \alpha$, each including all the substitutions of the group that do not involve a given element. The necessary and sufficient condition that the group is primitive is that each of these subgroups is maximal. If this condition is satisfied, $\alpha = 1$; but the converse is not always true, i. e. α may be equal to unity in an imprimitive and also in an intransitive group. The following method of constructing all the primitive groups of a small degree (n) is based very largely upon the given maximal subgroup of degree $n-1$, which is supposed to be known. We shall first consider the case when n is a prime number (p).

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[*] Cf. Quarterly Journal of Mathematics, vol. XXVIII, 1896, p. 207.

Every transitive group of degree p is primitive and contains at least one subgroup of order p. If it contains only one such subgroup, it must be included in the metacyclic group of this degree. All the transitive subgroups of this group are self-conjugate and correspond to the subgroups contained in the cyclical group of order $p-1$,[*] if we include identity and the entire group under the term subgroup. As a cyclical group contains one and only one subgroup whose order is any given divisor of the order of the cyclical group, there are just as many transitive groups of degree p that contains only one subgroup of order p as there are different divisors of $p-1$. E. g. there are just four primitive groups of degree seven that contain only one subgroup of order 7. The orders of these groups are 7, 14, 21 and 42.

All the subgroups of order p of any primitive group of degree p generate a self-conjugate subgroup, which is a simple group.[†] This simple group cannot be self-conjugate in more than one group of degree p and of a given order, since the substitutions which transform a subgroup of order p into itself are completely determined by this subgroup. Hence it is very easy to determine all the primitive groups of degree p that contain a given simple group as self-conjugate subgroup. We have only to find the largest subgroup of the metacyclic group that contains any one of the substitutions of order p which is found in the given simple group and transforms this simple group into itself. If the order of this subgroup is α times the order of the subgroup of the same metacyclic group that is contained in the given simple group, then this simple group is self-conjugate in just as many groups of degree p as there are different divisors of α, and the generators of these groups are contained in the given simple group and the given subgroup of the metacyclic group. Our problem is therefore reduced to the determination of the simple groups of degree p whose order exceeds p.

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[*] Since p has primitive roots.

[†] Miller, Bulletin of the American Mathematical Society, vol. IV (1898), p. 189.

Each subgroup of order p that is contained in such a group must be transformed into itself by more than p and by less than $p\,(p-1)$ substitutions of the group. Hence the order of the required groups must be of the form

$$\alpha p\,(1+kp), \qquad 1<\alpha<p-1$$

α must also be a divisor of p — 1. Sylow has recently considered the case when $k=1$, and he demonstrated that the groups of degrees five, seven, eleven and of orders 60, 168, 660 respectively are the only groups of a prime degree whose order is $\frac{1}{2}p\,(p^2-1)$.[*]

If a primitive group (G) of degree n is only simply transitive, its maximal subgroup (G) of degree $n-1$ is intransitive, and if the order of one of its transitive constituents is divisible by a given prime number (p), the order of each of the other transitive constituents is divisible by the same prime number.[†] Hence we observe that the order of G_1 must be a prime number whenever one of its transitive constituents is of a prime order. In particular, if G_1 contains a transitive constituent of degree 2, the order of G_1 is 2 and the degree of G is a prime number,[‡] i. e. G is contained in the metacylic group.

Since G_1 is a maximal group, none of its self-conjugate subgroups besides identity can be transformed into itself by any substitution that is not found in G_1. Hence we see that when G_1 contains a self-conjugate subgroup of (H) of degree $n-\alpha$, H must be intransitive and it must be transformed by substitutions of G into $\alpha-1$ other subgroups of G_1. When G_1 is Abelian or Hamiltonian, all its substitutions except identity must therefore be of degree $n-1$, and the order of G cannot exceed $\frac{n\,(n-1)}{2}$.[‖] These theorems will be very useful in considering what intransitive groups of degree $n-1$ could possibly occur in simply transitive primitive groups of degree n.

When G is k times transitive, its G_1 must be $k-1$ times transitive. When $n=p+\lambda$ or $2p+\lambda$, $\lambda>2$, G cannot be more than λ times transitive without being either the symmetric or the alternating group of degree n.[§] It is well known that G includes the alternating group of degree n whenever it involves a transposition or a cyclical substitution of degree three. If two substitutions have only one common element, their commutator is a cyclical substitution of degree three. Hence G cannot include two such substitutions without including the alternating group.

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[*] Sylow, "Videnskabs-Selskabets Skriften," 1897, No. 9.

[†] Jordan, "Traité des Substitutions," 1870, p. 284.

[‡] Miller, Proceedings of the London Mathematical Society, vol. XXVIII, p. 536.

[‖] Ibid., p. 584.

[§] Jordan, Bulletin de la Société Mathématique de France, vol. I, p. 41. Cf. Miller, Bulletin of the American Mathematical Society, vol. IV (1898), p. 142.

We observed that G_1 is a maximal subgroup of G. It evidently does not include any self-conjugate subgroup of G besides identity. Conversely, it is easy to prove that if a group of order g contains a maximal subgroup of order g which does not include any self-conjugate subgroup of the entire group, with the exception of identity, then this entire group may be represented as a primitive substitution-group of degree $g \div g_1$ and the number of ways in which it can be represented as such a primitive group may be directly determined from its abstract group properties.[*] We can also determine how many times such a primitive group is transitive from its abstract group properties.[†]

If G_1 is intransitive, every substitution of G that is not contained in G_1 must transform more than $G \div n(n-1)$ substitutions of G_1 into substitutions of G_1, g being the order of G. If it is transitive, every substitution of G that is not included in G_1 must transform just $g \div n(n-1)$ substitutions of G_1 into substitutions of this subgroup. Similar remarks evidently apply to G_1 and its subgroups in case G is more than doubly transitive.

PART II.

Determination of the Groups whose Degree does not exceed Eight.

By means of the methods given in Part I, it is easy to determine all the substitution-groups whose degree does not exceed eight. It is, however, not to be inferred that these methods are sufficient to determine all the groups of large degrees with a reasonable amount of labor. The difficulty of the problem to determine all the possible groups of degree n increases very rapidly with the increase of n, and little progress has been made towards its complete solution. The theorems of Cauchy, Sylow and Frobenius, which relate to the existence of certain subgroups in all the abstract groups of a given order and give some general conditions which the number of these subgroups must satisfy, are probably the most important steps in this direction.

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[*] Dyck, Mathematische Annalen, vol. XXII, p. 90. Cf. Miller, Bulletin of the American Mathematical Society, vol. III, p. 218.

[†] Messenger of Mathematics, 1898, pp. 104–107.

The theorems of Jordan, Bochert and Maillet, which relate to the class of primitive substitution-groups, are also of considerable general importance. A vast number of more special theorems have been published during recent years. Some of these will be very useful in the considerations which follow. In considering the groups of degree n we shall confine our attention to those which actually involve n letters. We shall thus give each substitution-group that does not involve more than eight letters once and only once. The notation explained by Cayley, Quarterly Journal of Mathematics, vol. XXV, will be employed throughout.

§1.—*Determination of the Groups whose Degree does not exceed Five.*

If we have only two letters, it is evident that the only possible permutation is obtained by interchanging the letters. Two such permutations performed in succession bring the letters to their original positions. If the letters are represented by a, b, these operations may be designated by ab and $(ab)^2 = 1$ respectively. They evidently form the only possible group of degree two. This group is designated by (ab).

The symmetric group of degree three is of order 6 and will be represented by (abc) all,* while the alternating group of this degree is of order 3 and will be represented by (abc) cyc. There can be no other transitive group of this degree because the order of a transitive group must be divisible by its degree, and there is only one group of degree n whose order is equal to that of the symmetric group of this degree, since this includes all the possible substitutions of degree n.† Since the symmetric group of degree n contains only one subgroup (the alternating group) whose order is obtained by dividing the order of the symmetric group by 2,‡ there can be only one group of degree three and of order 3. From the fact that a system of intransitivity must involve at least two elements, it follows directly that there can be no intransitive group of degree three.

An intransitive group of degree four must contain two transitive constituents of degree two. Hence there are two such groups. The first is obtained by forming the direct product of (ab) and (cd). It is of order 4, and will be

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* See explanations, p. 385.

† Abatti, "Memorie della società italiana delle scienze," t. 10, 1803, p. 385.

‡ Jordan, "Traité des substitutions" (1870), p. 66.

Since the divisors of 4 are 1, 2, 4, there are just three groups of degree five that contain one and only one subgroup of order 5.[*] The orders of these groups are 5, 10, 20. The first is cyclical, the second is semi-metacyclic, and the last is the metacyclic group of degree five. These groups are denoted by $(abcde)$ cyc, $(abcde)_{10}$ and $(abcde)_{20}$ respectively. There cannot be any other transitive group of degree five besides the alternating and the symmetric group, for such a group has to contain six subgroups of order 5 according to Sylow's theorem, and each of these subgroups must be transformed into itself by at least ten substitutions.[†] Hence the order cannot be less than 60.

§2.—*Determination of the Groups of Degree Six.*

If such a group is intransitive, the degrees of its transitive constituents must be one of the three sets 2, 2, 2 : 3, 3 : 4, 2. The largest group, all of whose transitive constituents are of degree two, is $(ab)(cd)(ef)$, and there is evidently only one such group. Since this contains only four positive substitutions, there is only one positive group of order 4 that contains three systems of intransitivity. As the average number of letters in all the substitutions of such a group is 3, a group of order 4 that involves negative substitutions must contain a transposition. This transposition and the positive substitution involving the other four elements must generate the group of order 4. Hence, there are two and only two groups of order 4 and degree six that contain three systems of intransitivity. There is evidently only one group of order 2 that contains these three systems. We have now found the four possible groups of degree six that involve the systems 2, 2, 2.

Since (abc) cyc and (def) all do not have the same quotient group, there is only one group that contains them as transitive constituents, viz. their direct product (abc) cyc (def) all. The two groups whose transitive constituents are (abc) cyc and (def) cyc are their direct product (abc) cyc (def) cyc and the group obtained by making them simply isomorphic $(abc.def)$ cyc. If the transitive constituents are (abc) all and (def) all, we may form their direct product (abc) all (def) all, their simple isomorphism $(abc.def)$ all, and a 3, 3 correspondence between them $\{(abc)$ all (def) all$\}$ pos. Since (abc) all is a complete group,[‡] we obtain only one group by making it simply isomorphic to itself.

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[*] Cf. Quarterly Journal of Mathematics, vol. XXIX, p. 226.

[†] See p. 308.

[‡] Hölder, Mathematische Annalen, vol. XLVI (1895), p. 325.

Hence, there are just six groups whose transitive constituents are 3, 3. All of them are evidently distinct abstract groups.

If the transitive constituents are of degrees four and two, we obtain five groups by forming the direct product of the transitive groups of degree four and (ef). The orders of these five groups [$(abcd)$ all (ef), $(abcd)$ pos (ef), $(abcd)_8(ef)$, $(abcd)_4(ef)$, $(abcd)$ cyc (ef)] are 48, 24, 16, 8, 8 respectively. We have seen that $(abcd)$ all and $(abcd)$ cyc contain one self-conjugate subgroup which includes one-half of the operators of the group, and that $(abcd)_8$ contains three such subgroups that cannot be transformed into each other while the three subgroups of order 2 in $(abcd)_4$ are transformed into each other by substitutions that transform $(abcd)_4$ into itself. Hence we obtain the following six groups by dimidiation: $\{(abcd)$ all $(ef)\}$ pos, $\{(abcd)$ cyc $(ef)\}$ pos, $\{(abcd)_4(ef)\}$ dim, and $\{(abcd)_8(ef)\}$ dim. The last symbol includes three groups which are simply isomorphic to $(abcd)_8$. Hence, there are 11 intransitives groups of degree six involving the systems 4, 2.

The imprimitive groups of degree six* have either two or three systems of imprimitivity. We shall first consider those groups which contain two such systems and afterward those which contain three systems without containing also two systems. From the general theory of Part I, it follows directly that each of these groups must contain one of the following five groups:

(abc) all (def) all, $\{(abc)$ all (def) all$\}$ pos, $(abc\,.\,def)$ all,
(abc) cyc (def) cyc, $(abc\,.\,def)$ cyc,

and that we obtain an imprimitive group whose order is twice the order of the head by adding a substitution of order two $(ad\,.\,be\,.\,cf)$ to each one of these heads. From theorems III and and V it follows directly that there is only one group for each of the three heads (abc) all (def) all, $(abc\,.\,def)$ all, (abc) cyc (def) cyc; and from theorem VI it follows that there are just two groups with the head $(abc\,.\,def)$ cyc. The orders of these five imprimitive groups are evidently 72, 12, 18, 6, 6 respectively. From theorem IV we observe that there are just two such groups that contain $\{(abc)$ all (def) all$\}$ pos. It is evident that these seven possible imprimitive groups of degree six, which contain two systems of imprimitivity, represent seven distinct abstract groups.

If an imprimitive group of degree six contains three systems of imprimitivity, it must have one of the following four groups as a head, and its order must be the order of its head multiplied by three or six:

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* Cf. Burnside, "Theory of Groups" (1897), p. 180.

$$1, \quad (ab\,.\,cd\,.\,ef), \quad \{(ab)(cd)(ef)\}\ \text{pos}, \quad (ab)(cd)(ef).$$

If it has either of the first two heads, its order must be 6 or 12. We observed above that the two possible regular groups of order 6 contain two systems of imprimitivity. It is evident that each of them contains also three systems, but we desire to consider only those groups which contain three systems without also containing two systems. The five possible groups of order 12 are well known;[*] and only two of them contain subgroups of order 2 that are not self-conjugate. As all these subgroups can be made to correspond in some simple isomorphism of the group to itself, and as they are not maximal, there are just two imprimitive groups of degree six and order 12. We found above the one that contains a self-conjugate subgroup of order 2; the one that is simply isomorphic to $(abcd)$ pos must, therefore, contain the third of the four heads given above, and it must permute its three systems according to the group of order 3.

Since all the operation-groups whose order is less than 64 are known,[†] we could readily find all the possible imprimitive groups that contain the given heads directly from the simply isomorphic operation-groups. It may, however, be more simple to proceed as follows: From theorem III, Part I, it follows that all the possible imprimitive groups of degree six which involve three systems of imprimitivity, must be contained in the group of order 48 which contains the head $(ab)(cd)(ef)$ and permutes its systems of intransitivity according to the symmetric group of degree three. It remains only to consider the subgroups of order 24 that are contained in this group. Since this group contains only one subgroup of order 12 and is isomorphic to the four-group with respect to this subgroup, it contains just three subgroups of order 24. That each of these three groups contains three systems of imprimitivity follows directly from the fact that the given group of order 48 permutes the systems of intransitivity of the given head according to the symmetric group of degree three. From the same fact we observe that these three groups of order 24 are distinct substitution-groups. The two which involve only the positive substitutions of $(ab)(cd)(ef)$ are simply isomorphic to $(abcd)$ all, while the one which contains $(ab)(cd)(ef)$ is the direct product of the alternating group of degree four and an operator of order 2.

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[*] Cayley, American Journal of Mathematics, vol. XI, pp. 139–157; cf. Kempe, Philosophical Transactions, vol. CLXXVII, pp. 37–43.

[†] Miller, Quarterly Journal of Mathematics, vol. XXIX, pp. 243–263.

We have now considered all the possible imprimitive groups of degree six and found that there are seven such groups that contain two systems of imprimitivity, and that there are five that contain three systems without containing also two systems, while there are just three groups that contain both two and three systems, viz. the two regular groups and one of the groups of order 12. The two given groups of order 24 are the only instances where two imprimitive groups of degree six represent the same abstract group.

Mathieu proved* that there is at least one triply transitive group of degree $p+1$ and order $p(p^2-1)$ and one doubly transitive group [the group of the modular equation] of degree $p+1$ and order $\frac{p}{2}(p^2-1)$, p being any prime number. The latter is simple whenever p exceeds 3, and it is formed by the positive substitutions of the former. It will appear that there is no other primitive group of degree six besides the alternating and the symmetric group.

Since each of the intransitive groups of degree five contains a substitution of the form abc, there can be no primitive group of degree six unless its order is divisible by 5. If such a group does not contain the alternating group, its order must be 30, 60 or 120. It could not be of order 30, since every group of this order contains a subgroup of order 15,† and a group of order 15 must be cyclical since $5-1$ is not divisible by 3. The only one of the thirteen possible groups of order 60 that contains six subgroups of order 5 is simply isomorphic to $(abcde)$ pos.‡ As the latter contains only one system of subgroups of order 12 that can be made to correspond, it can be represented in only one way as a transitive group of degree six, and this must therefore be the group mentioned above. If there is a primitive group of order 120, it must contain the given group of order 60 as subgroup, and it must be completely determined by this subgroup, since $(abcde)_{20}$ is completely determined by $(abcde)_{10}$. Hence, we have determined all the possible groups of degree six.

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* Journal de Mathematiques, vol. V (1860), p. 87.

† Frobenius, Berliner Sitzungsberichte, 1895, p. 1048; cf. Hölder, Göttingen Nachrichten, 1895, p. 211.

‡ Burnside, "Theory of Groups of a Finite Order," 1897, p. 107.

§3.—*Determination of the Groups of Degree Seven.*

Since seven is odd, one of the transitive constituents of each intransitive group of this degree must be either of degree three or of degree five. In the former case the other constituent may be either transitive or intransitive. Hence, every intransitive group of degree seven must have one of the following three sets of systems of intransitivity: 3, 4; 3, 2, 2; 5, 2. Since there are five transitive groups of degree four and two of degree three, we can construct just ten intransitive groups by forming the direct product of a transitive group of degree three and one of degree four. (*abcd*) all is the only group of degree four that has a non-cyclical quotient group of order 6, and (*abcd*) pos is the only one of these groups that has a quotient group of order 3; hence we obtain just two groups by establishing an α, 1 isomorphism between the transitive groups of degree four and the groups of degree three. The only other quotient group that is represented by the groups of degree three is of order 2, and we have seen that this gives rise to six groups. Hence, there are just eighteen intransitive groups of degree seven whose transitive constituents are of degrees three, four.*

Since there are only two intransitive groups of degree four, we obtain only four groups by forming the direct products of the intransitive groups of degree four and the groups of degree three. The only quotient group which these constituent groups have in common is of order 2. From the fact that (*ab*)(*cd*) contains two subgroups of order 2 that cannot be transformed into each other, it follows that we obtain three intransitive groups by dimidiating (*abc*) all and the intransitive groups of degree four. Hence there are just seven groups of degree seven whose transitive constituents are of degrees 3, 2, 2.

By multiplying (*fg*) into the five transitive groups of degree five, we obtain the five direct products whose transitive constituents are of degrees five and two. Since three of these transitive groups have one and only one subgroup of half their order, we obtain three additional intransitive groups by dimidiation. Hence, there are eight intransitive groups of degree seven whose transitive constituents are of degrees five and two, and the total number of intransitive groups of degree seven is thirty-three.

It follows directly from Part I that there are just four primitive groups of degree seven which contain only one subgroup of order 7, and that the orders of these groups are 7, 14, 21 and 42 respectively. The substitutions of order 7 that are contained in any other primitive group of this degree must generate a self-conjugate simple group of a composite order. The lowest order of a simple group that contains operators of order 7 is 168, and there is only one simple group of this order.* As all the subgroups of order 24 that are found in this simple group can be made to correspond in some simple isomorphism of the group to itself, it can be represented in only one way as a transitive group of degree seven.

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*Cf. Burnside, "Theory of Groups of a Finite Order," 1897, p. 165.

This simple group cannot occur as a self-conjugate subgroup of a larger primitive group of degree seven, since the groups of order 48 and degree six contain a transposition.† If other primitive groups of degree seven exist, they must include some other simple group of composite order as a self-conjugate subgroup. The only groups of degree six that could possibly occur in such a simple group are $(abcdef)_{60}$ and $(abcdef)$ pos. The former is impossible because there is no simple group of order 420,‡ and the latter leads to the alternating group of degree seven. Hence there is one and only one primitive group of each of the orders 7, 14, 21, 42, 168, $\frac{1}{2}7!$, $7!$

§4.—*Determination of the Groups of Degree Eight.*

An intransitive group of degree eight must have one of the following five sets of systems of intransitivity: 2, 2, 2, 2; 3, 3, 2; 3, 5; 4, 2, 2; 4, 4; 6, 2. The first of these sets was considered in Part I, where we proved that there are just eight groups of this kind. Those of the second set may be directly obtained from the groups of degree six whose systems of intransitivity are 3, 3. As there are six such groups of degree six, we obtain six groups of degree eight as the direct products of these groups of degree six and a transposition. Two of the groups of degree six, [$(abc.def)$ cyc, (abc) cyc (def) cyc], do not contain a quotient group of order 2; three others, [$(abc.def)$ all, (abc) all (def) cyc, {(abc) all (def) all} pos], contain a single group whose order is half the order of the group, while the last, (abc) all (def) all, contains three subgroups of order 18. As two of the last three subgroups are transformed into each other by substitutions that transform (abc) all (def) all into itself, there are just eleven intransitive groups of degree eight which involve the systems 3, 3, 2.

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* Hölder, Mathematische Annalen, vol. XL (1892), p. 83.

† Cayley, Quarterly Journal of Mathematics, vol. XXV, 1891, p. 81.

‡ Cole, American Journal of Mathematics, vol. XV (1893), p. 303.

By forming the direct products of the transitive groups of degree five and those of degree three, we obtain ten groups whose systems of intransitivity are 5, 3. As none of the quotient groups of the transitive groups of degree five is of order 3 or 6, we cannot form any groups by establishing an α, 1 isomorphism between the transitive groups of degree five and those of degree three. Hence, it is only necessary to add the three groups obtained by dimidiating some transitive group of degree five and the symmetric group of degree three in order to get the total number (13) of the groups of degree eight whose transitive constituents are of degrees five and three.

If we multiply each of the five transitive groups of degree four into the two intransitive ones, we obtain ten groups whose transitive constituents are of degrees four, two, two. By dimidiating the same groups we obtain eighteen additional groups with the given transitive constituents, since $(ab)(cd)$ has two subgroups of order 2 that cannot be transformed into each other. By establishing an α, 1 isomorphism between the transitive groups of degree four and $(ab)(cd)$, we obtain three additional groups from $(efgh)_8$ and one from $(efgh)_4$. Hence, the total number of these groups whose transitive constituents are 4, 2, 2 is 32. It is evident that the groups obtained by these different methods are distinct.

We obtain fifteen groups by forming the direct products of the transitive groups of degree four into themselves written in a different set of letters. Each of the transitive groups of degree four is transformed into all its simple isomorphisms to itself by the symmetric group of this degree,* with the exception of $(abcd)_8$. Since the group of isomorphism of the last group is simply isomorphic to the group itself, and it has three conjugates in $(abcd)$ all, we obtain just two distinct substitution-groups by making it simply isomorphic to itself. Hence, we obtain six additional groups by establishing a simple isomorphism between two transitive groups of degree four. We proceed to find the remaining groups whose transitive constituents are 4, 4.

The quotient groups of $(abcd)$ all which differ from the entire group are the group of order 2 and the symmetric group of order 6. We can evidently use the latter of these two only in establishing a 4, 4 correspondence between $(abcd)$ all and $(efgh)$ all and thus forming a group of order 96. The former can be used with all the groups that have a quotient-group of half their order, and hence leads to six additional groups. We have now found all the groups whose systems are 4, 4 and which contain the symmetric group as one of their transitive constituents, in addition to those which are obtained by forming the direct product or by establishing a simple isomorphism. We shall next find all the additional groups which have $(abcd)$ pos for one of their transitive constituents, then we shall find those which have $(abcd)_8$ as a transitive constituent without containing either $(abcd)$ all or $(abcd)$ pos, etc.

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* Cf. Hölder, Mathematische Annalen, vol. XLVI (1895), p. 340; Miller, Bulletin of the American Mathematical Society, vol. I (1895), p. 258.

Since $(abcd)$ pos contains only one self-conjugate subgroup besides identity, and this corresponds to a quotient-group of order 3, we obtain only one additional group that contains this group for one of its transitive constituents, viz. the group of order 48 which is formed by establishing a 4, 4 correspondence of the alternating group to itself written in a different set of letters. When one of the transitive constituents is $(abcd)_8$, we have to consider two quotient groups, viz. the four-group and the group of order 2. The former gives rise to five additional groups of order 16[*] and one of order 8, while the latter leads to six additional groups of order 32[†] and six of order 16. Hence, there are eighteen groups that contain $(abcd)_8$ as a transitive constituent but do not contain any transitive constituent of a larger order, and are neither the direc- product of two transitive groups of degree four nor the groups obtained by making $(abcd)_8$ simply isomorphic to itself. Since we obtain three additional groups by dimidiating $(abcd)_4$ and $(abcd)$ cyc written in two systems of letters, the total number of the groups of degree eight whose transitive constituents are 4, 4 is 50.

By multiplying each of the sixteen transitive groups of degree six by a transposition we obtain sixteen intransitive groups whose transitive constituents are 6, 2. It remains to examine all the transitive groups of degree six in regard to their subgroups whose orders are obtained by dividing the order of the group by 2 and which cannot be transformed into each other by any substitution that transforms the group into itself. It is well known that each of the groups of order 6 and of order 18 contains only one such subgroup. One of the groups of order 12 contains three such subgroups, while the other does not contain any subgroup of order 6. We have already observed that each of the three transitive groups of order 24 contains only one subgroup of order 12, and that the group of order 48 contains three distinct subgroups of order 24. The group of order 36 which contains substitutions of order 4 contains only one subgroup of order 18, since it is isomorphic to the cyclical group of order 4 with regard to its subgroup of order 9, while the other group of order 36 contains three subgroups of order 18, two of which are conjugate in the group of order 72. Since the group of order 72 is isomorphic to $(abcd)_8$ with respect to its subgroup of order 9, it contains just three subgroups of order 36—the two transitive groups just mentioned and the direct product of two symmetric groups of degree three. Hence, we obtain eighteen groups by dimidiating the imprimitive groups of degree six and (gh).

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[*] Cole, Bulletin of the New York Mathematical Society, vol. II (1893), p. 187; cf. Miller, ibid., vol. III (1894), p. 168.

[†] Cayley, Quarterly Journal of Mathematics, vol. XXV (1891), p. 147.

Since the four primitive groups of degree six (regarded as abstract groups) are the symmetric and the alternating groups of degrees five and six, we obtain only two groups by dimidiating a primitive group of degree six and (gh). Hence, there are thirty-six groups of degree eight whose transitive constituents are of degrees six, two. The total number of intransitive groups of this degree is therefore $8 + 11 + 13 + 32 + 50 + 36 = 150$. It remains to determine the transitive groups.

The imprimitive groups of degree eight contain either two or four systems of imprimitivity. We shall first consider those which contain two such systems and afterwards those which contain four systems without containing also two systems. The following twenty-three groups may be used as heads: $(abcd)$ all $(efgh)$ all, $\{(abcd)$ all $(efgh)$ all$\}$ pos, $\{(abcd)$ all $(efgh)$ all$\}_{4,4}$, $(abcd \,.\, efgh)$ all, $(abcd)$ pos $(efgh)$ pos, $\{(abcd)$ pos $(efgh)$ pos$\}_{4,4}$, $(abcd \,.\, efgh)$ pos, $(abcd)_8 (efgh)_8$, $\{(abcd)_8 (efgh)_8\}$ dim (three groups), $\{(abcd)_8 (efgh)_8\}_{2,2}$ (four groups), $(abcd \,.\, efgh)_8$ (two groups), $(abcd)_4 (efgh)_4$, $\{(abcd)_4 (efgh)_4\}$ dim, $(abcd \,.\, efgh)_4$, $(abcd)$ cyc $(efgh)$ cyc, $\{(abcd)$ cyc $(efgh)$ cyc$\}$ pos, $(abcd \,.\, efgh)$ cyc.

We proved in Part I that there are just five distinct imprimitive groups of degree eight in which the transitive constituents of the head are the symmetric groups of degree four, and that there are five such groups in which the transitive constituents of the head are the alternating groups of degree four. Hence, the first seven of the given twenty-three heads give rise to ten imprimitive groups of degree eight. Three of the remaining heads are the direct products of two transitive groups, and, hence, each gives rise to only one imprimitive group. It is evident that these three groups are distinct from each other and from those which involve the other heads.

From theorem IV it follows that each of the three heads represented by $\{(abcd)_8\,(efgh)_8\}$ dim gives rise to two groups, and that these six groups of order 64 are distinct. From the same theorem we observe that the group represented by $\{(abcd)_8\,(efgh)\}_{2,\,2}$ in which the identical divisions correspond, is the head of four distinct groups of order 32, and also that each of the other three heads represented by this notation leads to two conjugate groups. Two of the last three groups involve substitutions of degree six, and must therefore be distinct from the other groups of order 32 which have been given above; the third contains a subgroup of order 16, which is conjugate to the $\{(abcd)_8\,(efgh)_8\}_{2,\,2}$ in which the identical divisions correspond, and therefore is conjugate to one of the four groups containing this head. Hence, there are just six distinct groups that contain one of the four heads represented by $\{(abcd)_8\,(efgh)_8\}_{2,\,2}$.[*]

As the remaining heads lead to groups of order 8 or 16, it may be easiest to obtain them from their simply isomorphic abstract groups. Since each abstract group can be represented in just one way as a regular group, there are five regular groups of degree eight, and their substitutions[†] can be directly obtained from the simply isomorphic abstract groups. It remains to consider the groups of order 16 which can be represented as transitive groups of degree eight, and to determine in how many ways each group can be so represented. Since an Abelian or a Hamiltonian group cannot be transitive unless it is regular, we may confine our attention to the eight groups of order 16 that are neither Abelian nor Hamiltonian. A list of these groups, written out in full, is given in Quarterly Journal of Mathematics, vol. XXVIII, pp. 269–273.

It is evident that the groups Nos. 6 and 11 in this list do not contain any subgroup of order 2 that is not self-conjugate, and that each of the remaining six groups contains such subgroups. In at least five of these groups all of these subgroups of order 2 can be made to correspond in some simple isomorphism of the group to itself, since they do not occur in one of the Abelian subgroups of order 8; hence, each of these groups can be represented in only one way as a transitive substitution group of degree eight. No. 9 contains three Abelian subgroups of order 8, each containing two subgroups of order 2 that are not self-conjugate in the entire group. From the group of isomorphisms of this group,[‡] it follows directly that all of these subgroups of order 2 correspond in some simple isomorphism of the group to itself. Hence, it also can be represented in only one way as a transitive substitution group of degree eight. From this we see that there are just six transitive substitution-groups of degree eight and order 16. These must be imprimitive, since a group of order p^a, p being any prime number and $\alpha > 1$, cannot be primitive. They must contain two systems of imprimitivity, since they could not be isomorphic to a primitive group of degree four.

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[*] Cole, Bulletin of the New York Mathematical Society, vol. II (1893), p. 188.

[†] Cayley, Quarterly Journal of Mathematics, vol. XXV (1891), p. 148.

[‡] Miller, Quarterly Journal of Mathematics, vol. XXVIII, p. 268.

We have now considered all the possible cases and found that there are just thirty-six groups of degree eight that contain two systems of imprimitivity. It remains to determine those which contain four systems without also containing two systems. Since such groups must permute their systems according to $(abcd)$ pos or $(abcd)$ all, the systems of intransitivity of the heads must permit the permutations represented by these groups. Hence, we need to consider only the following heads:

$$1, \quad (ab\,.\,cd\,.\,ef\,.\,gh), \quad \{(ab)(cd)(ef)(gh)\}\text{ pos}, \quad (ab)(cd)(ef)(gh).$$

The first one of these heads does not lead to an additional group, since $(abcd)$ all can be represented in only one way as a transitive substitution-group of degree eight, having only one system of conjugate subgroups of order 3. It is evident that one such representation is possible with the head $(abcd\,.\,efgh)$ pos. The second of these heads gives rise to groups of order 24 and 48 which contain non-maximal subgroups of orders 3 and 6 respectively that are not self-conjugate nor contain any self-conjugate subgroup of the entire group. If a transitive group of order 24 and degree eight contains a subgroup of order 12, this must be intransitive and the group must contain two systems of imprimitivity. Since there is only one group of order 24 that does not contain a subgroup of order 12* and its subgroups of order 3, are non-maximal and conjugate by Sylow's theorem, there is only one imprimitive group of order 24 and degree eight that does not have two systems of imprimitivity.

The groups of order 48 that can be represented as transitive groups of degree eight, must clearly be contained among the eight groups of this order that contain four subgroups of order 3.† Four of these eight groups contain a self-conjugate subgroup of order 12, and hence they must contain two systems of imprimitivity if they can be represented as transitive groups of degree eight. In fact, only one of these four groups (G_{47}) can be represented as a transitive group of degree eight. It is also easily seen that only one of the remaining four groups (G_{51}) contains non-maximal subgroups of order 6 that do not include any operator except identity that is self-conjugate, and that all such subgroups may be made to correspond in some simple isomorphism of G_{51} to itself. Hence, there is only one imprimitive group of order 48 that contains four systems of imprimitivity without containing also two systems. That there are only two groups with the head $(ab)(cd)(ef)(gh)$ and three with the head $\{(ab)(cd)(ef)(gh)\}$ pos following directly from the general theory. Hence there are just seven groups of degree eight that contain four systems of imprimitivity without containing also two such systems. The total number of imprimitive groups of this degree is therefore forty-three.

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*Miller, Quarterly Journal of Mathematics, vol. XXVIII, p. 274.

† Ibid., vol. XXX, p. 258.

Each of the solvable primitive groups of degree eight must contain as a self-conjugate subgroup the regular group of order eight which contains no operator whose order exceeds 2. Since the group of isomorphisms of this regular group is $(abcdefg)_{168}$,[*] these primitive groups must correspond to subgroups of this group of order 168. It is evident that a doubly transitive group of order 56 corresponds to the subgroup of order 7, and that a doubly transitive group of order 168 corresponds to the subgroup of order 21. Since all of the subgroups of orders 7 and 21 are conjugate in $(abcdefg)_{168}$, there is only one solvable primitive group of each of the given orders.

If there were any other solvable primitive group of degree eight, it would have to be simply transitive, and its maximal subgroup of degree seven would have to be a subgroup of $(abcdefg)_{168}$[†] and contain the systems 3, 4.[‡] As this intransitive group could not contain any substitution whose degree is less than four nor a transitive group whose degree is less than six, it cannot be constructed. Hence, there are only two solvable primitive groups of degree eight.

From what precedes, it follows that the largest group of degree eight which contains the given regular group as a self-conjugate subgroup is triply transitive and of order 1344. The two Mathieu groups of orders $p(p^2-1)$, $\frac{p}{2}(p^2-1)$

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[*] Moore, Bulletin of the American Mathematical Society, vol. I (1894), p. 61.

[†] Ibid., vol. V (1899), p. 250.

[‡] Jordan, "Traité des substitutions" (1870), p. 284.

and degree $p + 1$ are of orders 336 and 168 respectively. Since the latter is simple, it is distinct from the solvable primitive group of this order which we determined above.* If we add to these groups the alternating and the symmetric, we have seven primitive groups of degree eight, two being solvable and five insolvable. We proceed to prove that there is no other primitive group of this degree.

If a primitive group of degree eight were simply transitive, its maximal subgroup of degree seven would have transitive constituents of degrees three and four. As it could not contain any transitive subgroup whose degree is less than eight† nor any substitution whose degree is less than four, such a group cannot be constructed.‡ Hence, it is only necessary to consider groups of orders 56, 112, 168, 336 and 1344. As we have considered all the possible solvable groups, we may suppose that the groups under consideration contain a composite factor of composition. Hence, we do not need to consider the first two of the given orders. We proved in Part I that there cannot be more than one group of degree eight and order 1344. We may, therefore, restrict ourselves to groups of orders 168 and 336.

Since there is only one simple group of order 168,§ and this group contains only one system of conjugate subgroups of order 21, it can be represented in only one way as a transitive group of degree eight. An insolvable group of order 336 must contain the given simple group as self-conjugate subgroup, and is completely determined by it since $(abcdefg)_{42}$ is completely determined by $(abcdefg)_{21}$. Hence the proof is complete. We have now examined all the possible cases and found that there are just two hundred substitution-groups of degree eight; one hundred and fifty of these are intransitive, forty-three are imprimitive and seven are primitive.

PART III.

List of all the Substitution-Groups whose Degree is less than Nine.

In the following list the groups which are simply isomorphic to groups which precede them, are represented by Roman letters, and the others are represented by Greek letters. All these substitution-groups, which are represented by Greek letters, are therefore distinct as abstract or operation-groups; and they include all the abstract groups which can be represented by substitution-groups whose degree does not exceed eight.

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*Cf. Hölder, Mathematische Annalen, vol. XL, p. 75.

†Cf. Netto, "Theory of Substitutions" (1892), p. 95.

‡Cf. Miller, Proceedings of the London Mathematical Society, vol. XXVIII, p. 534.

§Burnside, "Theory of Groups of a Finite Order," 1897, p. 208.

The number which follows a group represented by Roman letters indicates the degree of the first substitution-group (represented by Greek letters) to which it is simply isomorphic, and the subscript of this number indicates which of the groups of the given degree and order is meant. If there is only one such group, this subscript is omitted. E. g. the 3 after the second and third groups of degree six and order 6 indicates that these two groups are simply isomorphic to the group of degree three and order 6, while the 6_1 after the third group of degree six and order 8 indicates that this is simply isomorphic to the first group of this degree and order.

Each of the insolvable groups is followed by the abbreviation *ins.* It may be observed that only 28 of the 295 substitution-groups whose degree does not exceed eight are insolvable. The number of abstract groups represented by these 295 substitution-groups is 137, i. e. somewhat less than one-half of the total number. The number of abstract insolvable groups is 20; five of these are simple groups of composite order, viz. one group of each of the order 60, 168, 360, 2520 and 20160.

LIST.

Degree Two.

Order.	No.	Notation.	
2	1	$(\alpha\beta)$.	
		Degree Three.	
3	1	$(\alpha\beta\gamma)$ cyc,	
6	1	$(\alpha\beta\gamma)$ all.	
		Degree Four.	
2	1	$(ab\,.\,cd)$,	2
4	1	$(\alpha\beta)(\gamma\delta)$,	
	2	$(abcd)_4$,	4_1
	3	$(\alpha\beta\gamma\delta)$ cyc,	
8	1	$(\alpha\beta\gamma\delta)_8$,	
12	1	$(\alpha\beta\gamma\delta)$ pos,	
24	1	$(\alpha\beta\gamma\delta)$ all.	
Total,	7		

Degree Five.

Order.	No.	Notation.	
5	1	$(\alpha\beta\gamma\delta\varepsilon)$ cyc,	
6	1	$(\alpha\beta\gamma)$ cyc $(\delta\varepsilon)$	
	2	$\{(abc)$ all $(de)\}$ pos,	3
10	1	$(\alpha\beta\gamma\delta\varepsilon)_{10}$,	
12	1	$(\alpha\beta\gamma)$ all $(\delta\varepsilon)$,	
20	1	$(\alpha\beta\gamma\delta\varepsilon)_{20}$,	
60	1	$(\alpha\beta\gamma\delta\varepsilon)$ pos,	ins.
120	1	$(\alpha\beta\gamma\delta\varepsilon)$ all,	ins.
Total,	8		

Degree Six.

Order.	No.	Notation.	
2	1	$(ab\,.\,cd\,.\,ef)$,	2
3	1	$(abc\,.\,def)$ cyc,	3
4	1	$(ab\,.\,cd)(ef)$,	4_1
	2	$\{(ab)(cd)(ef)\}$ pos,	"
	3	$\{(abcd)_4(ef)\}$ dim,	"
	4	$\{(abcd)$ cyc $(ef)\}$ pos,	4_2
6	1	$(abcdef)$ cyc,	5
	2	$(abc\,.\,def)$ all,	3
	3	$(abcdef)_6$,	"
8	1	$(\alpha\beta)(\gamma\delta)(\varepsilon\zeta)$,	
	2	$(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta)$,	
	3	$(abcd)_4(ef)$,	6_1
	4	$\{(abcd)_8$ com $(ef)\}$ dim,	4
	5	$\{(abcd)_8$ cyc (ef) dim,	"
	6	$\{(abcd)_8$ pos $(ef)\}$ dim,	"
9	1	$(\alpha\beta\gamma)$ cyc $(\delta\varepsilon\zeta)$ cyc,	
12	1	$(abcdef)_{12}$,	5
	2	$(abcdef)_{12_2}$,	4
16	1	$(\alpha\beta\gamma\delta)_8(\varepsilon\zeta)$,	
18	1	$(\alpha\beta\gamma)$ all $(\delta\varepsilon\zeta)$ cyc,	
	2	$\{(\alpha\beta\gamma)$ all $(\delta\varepsilon\zeta)$ all$\}$ pos,	
	3	$(abcdef)_{18}$,	6_1

Order.	No.	Notation.		
24	1	$(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta)$,		
	2	$\{(abcd)$ all $(ef)\}$ pos,		4
	3	$(\pm\, abcdef)_{24}$,		"
	4	$(+\, abcdef)_{24}$,		"
	5	$(abcdef)_{24_2}$,		6_1
36	1	$(\alpha\beta\gamma)$ all $(\delta\varepsilon\zeta)$ all,		
	2	$(\alpha\beta\gamma\delta\varepsilon\zeta)_{36}$,		
	3	$(abcdef)_{36_2}$,		6_1
48	1	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta)$,		
	2	$(abcdef)_{48}$,		6_1
60	1	$(abcdef)_{60}$,	ins.	5
72	1	$(\alpha\beta\gamma\delta\varepsilon\zeta)_{72}$,		
120	1	$(abcdef)_{120}$,	ins.	5
360	1	$(\alpha\beta\gamma\delta\varepsilon\zeta)$ pos,	"	
720	1	$(\alpha\beta\gamma\delta\varepsilon\zeta)$ all,	"	
Total,	37			

Degree Seven.

6	1	$(ac\,.\,bd)(efg)$ cyc,	5
	2	$\{(ac\,.\,bd)(efg)$ all$\}$ dim,	3
7	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta)$ cyc,	
10	1	$(\alpha\beta\gamma\delta\varepsilon)$ cyc $(\zeta\eta)$,	
	2	$\{(abcde)_{10}(fg)\}$ dim,	5
12	1	$(ac\,.\,bd)(efg)$ all,	"
	2	$(\alpha\beta)(\gamma\delta)(\varepsilon\zeta\eta)$ cyc,	
	3	$(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta\eta)$ cyc,	
	4	$(abcd)_4(efg)$ cyc,	7_2
	5	$\{(ab)(cd)(efg)$ all$\}$ pos,	5
	6	$\{(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta\eta)$ all$\}$ pos,	
	7	$\{(abcd)_4(efg)$ all$\}$ dim,	5
	8	$\{(abc)$ all $(de\}$ pos (fg),	"
	9	$\{(abcd)$ pos (efg) cyc$\}$ tris,	4
14	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta)_{14}$,	
20	1	$(\alpha\beta\gamma\delta\varepsilon)_{10}(\zeta\eta)$,	

Order.	No.	Notation.		
	2	$\{(abcde)_{20}(fg)\}$ pos,		5
21	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta)_{21}$,		
24	1	$(\alpha\beta)(\gamma\delta)(\varepsilon\zeta\eta)$ all,		
	2	$(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta\eta)$ all,		
	3	$(abcd)_4(efg)$ all,		7_1
	4	$(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta)$ cyc,		
	5	$\{(\alpha\beta\gamma\delta)_8$ com $(\varepsilon\zeta\eta)$ all$\}$ dim,		
	6	$\{(\alpha\beta\gamma\delta)_8$ cyc $(\varepsilon\zeta\eta)$ all$\}$ dim,		
	7	$\{(abcd)_8$ pos (efg) all$\}$ dim,		7_5
	8	$\{(abcd)$ all (efg) all$\}_{4,1}$,		4
36	1	$(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta\eta)$ cyc,		
40	1	$(\alpha\beta\gamma\delta\varepsilon)_{20}(\zeta\eta)$,		
42	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta)_{42}$,		
48	1	$(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta)$ all,		
72	1	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta)$ cyc,		
	2	$(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta\eta)$ all,		
	3	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta)$ all$\}$ pos,		
120	1	$(\alpha\beta\gamma\delta\varepsilon)$ pos $(\zeta\eta)$,	ins.	
	2	$\{(abcde)$ all $(fg)\}$ pos,	ins.	5
144	1	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta)$ all,		
168	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta)_{168}$,	ins.	
240	1	$(\alpha\beta\gamma\delta\varepsilon)$ all $(\zeta\eta)$,	ins.	
2520	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta)$ pos,	ins.	
5040	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta)$ all,	ins.	
Total,	40			

Degree Eight.

2	1	$(ab.cd:ef.gh)$,	2
4	1	$(ab.cd.ef)(gh)$,	4_1
	2	$(ab.cd)(ef.gh)$,	"
	3	$\{(ab)(cd)(ef.gh)\}$ dim,	"
	4	$\{(abcd)$ cyc $(ef.gh)\}$ dim,	4_3
	5	$\{(abcd)_4(ef.gh)\}$ dim,	4_1
	6	$(abcd.efgh)$ cyc,	4_3

Order.	No.	Notation.	
	7	$\{(abcd)_4(ef)(gh)\}_{1,1}$,	4_1
	8	$(abcd\,.\,efgh)_4$,	"
6	1	$(abc\,.\,def)$ cyc (gh),	5
	2	$\{(abcdef)$ cyc $(gh)\}$ pos,	"
	3	$\{(abc\,.\,def)$ all $(gh)\}$ dim,	3
	4	$\{(abcdef)_6(gh)$ dim,	"
8	1	$(ab\,.\,cd)(ef)(gh)$,	6_1
	2	$\{(ab)(cd)(ef)\}$ pos (gh),	"
	3	$\{(abcd)$ cyc $(ef)\}$ pos (gh),	6_2
	4	$\{(abcd)_4(ef)\}$ dim (gh),	6_1
	5	$(abcd)$ cyc $(ef\,.\,gh)$,	6_2
	6	$(abcd)_4(ef\,.\,gh)$,	6_1
	7	$1,\ abcd,\ ac\,.\,bd,\ adcb + ef\,.\,gh\,(ac,\ bd,\ ab\,.\,cd,\ ad\,.\,bc)$,	
	8	$1,\ ab\,.\,cd,\ ac\,.\,bd,\ ad\,.\,bc + ef\,.\,gh\,(ac,\ bd,\ abcd,\ adcb)$,	
	9	$1,\ ac,\ bd,\ ac\,.\,bd + ef\,.\,gh\,(abcd,\ adcb,\ ab\,.\,cd,\ ad\,.\,bc)$,	
	10	$\{(ab)(cd)(ef)(gh)\}$ pos,	6_1
	11	$\{(abcd)$ cyc $(efgh)$ cyc$\}_{2,2}$,	6_2
	12	$\{(abcd)_4(efgh)_4\}_{2,2}$,	6_1
	13	$\{(ac)(bd)(efgh)$ cyc$\}$ pos,	6_2
	14	$\{(ac)(bd)(efgh)_4\}$ dim,	6_1
	15	$\{(abcd)_4(efgh)$ cyc$\}$ dim,	6_2
	16	$(abcd\,.\,efgh)_{8_1}$,	4
	17	$\{(abcd)_8(ef)(gh)\}_{2,1}$,	"
	18	$\{(abcd)_8(ef)(gh)\}_{2,1}$,	"
	19	$\{(abcd)_8(ef)(gh)\}_{2,1}$,	"
	20	$(abcd\,.\,efgh)\ 8_2$,	"
	21	$\{(abcd)_8(efgh)_4\}_{2,1}$,	"
	22	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta)$ cyc,	
	23	$A\,(abcdefgh)_8$,	6_1
	24	$B\,(abcdefgh)_8$,	4
	25	$C\,(abcdefgh)_8$,	6_2
	26	$D\,(\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta)_8$,	
12	1	$(abcdef)$ cyc (gh),	7_2
	2	$(abc\,.\,def)$ all (gh),	5

Order.	No.	Notation.	
	3	$(abcdef)_6(gh)$,	5
	4	$\{(abcdef)_{12}(gh)\}$ dim,	"
	5	$\{(abcdef)_{12}(gh)\}$ pos,	"
	6	$\{(abcdef)_{12}(gh)\}$ dim_2,	"
	7	$(abcd\,.\,efgh)$ pos,	4
15	1	$(\alpha\beta\gamma\delta\varepsilon)$ cyc $(\zeta\eta\theta)$ cyc,	
16	1	$(\alpha\beta)(\gamma\delta)(\varepsilon\zeta)(\eta\theta)$,	
	2	$(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta)(\eta\theta)$,	
	3	$(abcd)_4(ef)(gh)$,	8_1
	4–9	$\{(ab)(cd)(efgh)_8\}$ dim,	6
	10	$(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta\eta\theta)$ cyc,	
	11	$(abcd)_4(efgh)$ cyc,	8_2
	12	$(abcd)_4(efgh)_4$,	8_1
	13	$(abcd)_8(ef\,.\,gh)$,	6
	14–16	$\{(abcd)_4(efgh)_8\}$ dim,	"
	17, 18	$\{(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta\eta\theta)_8\}$ dim_1,	
	19	$\{(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta\eta\theta)_8\}$ dim_2,	
	20, 21	$\{(abcd)_8(efgh)_8\}_{2,2}$,	6
	22–24	$\{(abcd)_8(efgh)_8\}_{2,2}$,	8_{17}
	25	$\{(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta\eta\theta)$ cyc$\}$ pos $(\alpha\varepsilon\,.\,\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,	
	26	$\{(\alpha\beta\gamma\delta)$ cyc $(\varepsilon\zeta\eta\theta)$ cyc$\}$ pos $(\alpha\zeta\beta\eta\gamma\theta\delta\varepsilon)$,	
	27	$\{(abcd)_4(efgh)_4\}$ dim $(ae\,.\,bf\,.\,cg\,.\,dh)$,	6
	28	$\{(abcd)_4(efgh)_4\}$ dim $(afbe\,.\,chdg)$,	8_{17}
	29	$(\alpha\beta\gamma\delta\,.\,\varepsilon\zeta\eta\theta)_{8_2}(\alpha\varepsilon\beta\zeta\gamma\eta\delta\theta)$,	
	30	$(\alpha\beta\gamma\delta)\,.\,\varepsilon\zeta\eta\theta_{8_2}(\alpha\zeta\beta\eta\gamma\theta\delta\varepsilon)$,	
18	1	$(\alpha\beta\gamma)$ cyc $(\delta\varepsilon\zeta)$ cyc $(\eta\theta)$,	
	2	$\{(abc)$ all $(de)\}$ pos (fgh) cyc,	6_1
	3	$\{(abc)$ all (def) all$\}$ pos (gh) dim,	6_2
	4	$\{(abcdef)_{18}(gh)\}$ pos,	6_1
24	1	$(abcdef)_{12}(gh)$,	7_1
	2	$(abcd)$ pos $(ef\,.\,gh)$,	6_1
	3	$\{(abcd)$ all $(ef\,.\,gh)\}$ dim,	4
	4	$(abcd\,.\,efgh)$ all,	"
	5	$\{(+abcdef)_{24}(gh)\}$ dim,	"

Order.	No.	Notation.	
	6	$\{(\pm abcdef)_{24}(gh)\}$ pos,	4
	7	$\{(abcdef)_{24_2}(gh)\}$ pos,	6_1
	8	$(abcdef)_{12_2}(gh)$,	"
	9	$(abcd.efgh)$ pos $(ae.bf.cg.dh)$,	"
	10	$(\alpha\delta\varepsilon.\beta\gamma\zeta)(\alpha\zeta\beta\varepsilon.\gamma\theta\delta\eta)$,	
	11	$(abcd.efgh)$ pos $(af.be.cg.dh)$,	4
30	1	$(\alpha\beta\gamma\delta\varepsilon)$ cyc $(\zeta\eta\theta)$ all,	
	2	$(\alpha\beta\gamma\delta\varepsilon)_{10}(\zeta\eta\theta)$ cyc,	
	3	$\{(\alpha\beta\gamma\delta\varepsilon)_{10}(\zeta\eta\theta)$ all$\}$ dim,	
32	1	$(\alpha\beta\gamma\delta)_8(\varepsilon\zeta)(\eta\theta)$,	
	2	$(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)$ cyc,	
	3	$(abcd)_8(efgh)_4$,	8_1
	4	$L\{(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)_8\}$ dim,	
	5	$M\{(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)_8\}$ dim,	
	6	$N\{(abcd)_8(efgh)_8\}$ dim,	8_4
	7	$P\{(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)_8\}$ dim,	
	8	$Q\{(abcd)_8(efgh)_8\}$ dim,	8_4
	9	$R\{(abcd)_8(efgh)_8\}$ dim,	8_7
	10	$(\alpha\beta\gamma\delta)$ cyc) $(\varepsilon\zeta\eta\theta)$ cyc $(\alpha\varepsilon.\beta\zeta.\gamma\eta.\delta\theta)$,	
	11	$(abcd)_4(efgh)_4(ae.bf.cg.dh)$,	8_4
	12	$\{(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)_8\}_{2,2}(\alpha\varepsilon.\beta\zeta.\gamma\eta.\delta\theta)$,	
	13	$\{(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)_8\}_{2,2}(\alpha\varepsilon\beta\zeta.\gamma\eta\delta\theta)$,	
	14	$\{(abcd)_8(efgh)_8\}_{2,2}(aecg.bf.dh)$,	8_{13}
	15	$\{(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)_8\}_{2,2}(\alpha\varepsilon\beta\zeta\gamma\eta\delta\theta)$,	
	16	$A\{(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)_8\}_{2,2}(\alpha\varepsilon.\beta\zeta.\gamma\eta.\delta\theta)$,	
	17	$B\{(abcd)_8(efgh)_8\}_{2,2}(ae.bf.cg.dh)$,	8_{13}
36	1	$\{(abc)$ all $(de)\}$ pos (fgh) all,	6_1
	2	$\{(\alpha\beta\gamma)$ all $(\delta\varepsilon\zeta)$ all$\}$ pos $(\eta\theta)$,	
	3	$\{(abc)$ all (def) all $(gh)\}$ pos,	6_1
	4	$(\alpha\beta\gamma\delta\varepsilon\zeta)_{18}(\eta\theta)$,	
	5	$\{(abcdef)_{36}(gh)\}$ dim,	6_2
	6	$\{(abcdef)_{36_2}(gh)\}$ pos,	6_1
	7	$\{(abcdef)_{36_2}(gh)\}$ dim,	6_1
	8	(abc) all (def) cyc (gh),	8_4

Order.	No.	Notation.	
48	1	$(abcd)$ all $(ef\,.\,gh)$,	6_1
	2	$(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta)(\eta\theta)$,	
	3	$(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta\eta\theta)$ cyc,	
	4	$(abcd)$ pos $(efgh)_4$,	8_2
	5	$\{(abcd)$ all $(ef)\}$ pos (gh),	6_1
	6	$(\pm\, abcdef)_{24}\,(gh)$,	"
	7	$(+\, abcdef)_{24}\,(gh)$,	"
	8	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ cyc$\}$ dim,	
	9	$\{(abcd)$ all $(ef)(gh)\}$ dim,	6_1
	10	$\{(abcd)$ all $(efgh)_4\}$ dim,	"
	11–13	$\{(abcdef)_{48}\,(gh)\}$ dim,	"
	14	$\{(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta\eta\theta)$ pos$\}_{4,\,4}$,	
	15	$(abcdef)_{24_4}\,(gh)$,	8_2
	16	$(abcd\,.\,efgh)$ all $(ae\,.\,bf\,.\,cg\,.\,dh)$,	6_1
	17	$(\alpha\delta\varepsilon\,.\,\beta\gamma\zeta)(\alpha\gamma\zeta\theta\beta\delta\varepsilon\eta)$,	
56	1	$A\,(\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta)_8\,(\beta\gamma\varepsilon\delta\eta\theta\zeta)$ cyc,	
60	1	$(\alpha\beta\gamma\delta\varepsilon)_{10}\,(\zeta\eta\theta)$ all,	
	2	$(\alpha\beta\gamma\delta\varepsilon)_{20}\,(\zeta\eta\theta)$ cyc,	
	3	$\{(\alpha\beta\gamma\delta\varepsilon)_{20}\,(\zeta\eta\theta)$ all$\}$ pos,	
64	1	$(\alpha\beta\gamma\delta)_8\,(\varepsilon\zeta\eta\theta)_8$,	
	2	$M\{(\alpha\beta\gamma\delta)_8\,(\varepsilon\zeta\eta\theta)_8\}$ dim $(\alpha\varepsilon\,.\,\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,	
	3	$N\{(\alpha\beta\gamma\delta)_8\,(\varepsilon\zeta\eta\theta)_8\}$ dim $(\alpha\varepsilon\,.\,\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,	
	4	$\{(\alpha\beta\gamma\delta)_8\,(\varepsilon\zeta\eta\theta)_8$ cyc$\}$ dim $(\alpha\varepsilon\gamma\eta\,.\,\beta\zeta\,.\,\delta\theta)$,	
	5	$\{(\alpha\beta\gamma\delta)_8\,(\varepsilon\zeta\eta\theta)_8$ pos$\}$ dim $(\alpha\varepsilon\beta\zeta\gamma\eta\delta\theta)$,	
	6	$\{(abcd)_8\,(efgh)_8$ com$\}$ dim $(ae\,.\,bf\,.\,cg\,.\,dh)$,	8_3
	7	$\{(abcd)_8\,(efgh)_8$ com$\}$ dim $(aebfcgdh)$,	8_5
72	1	$(\alpha\beta\gamma)$ all $(\delta\varepsilon\zeta)$ all $(\eta\theta)$,	
	2	$(\alpha\beta\gamma\delta\varepsilon\zeta)_{36}\,(\eta\theta)$,	
	3	$(abcdef)_{36_3}\,(dh)$,	8_1
	4–6	$\{(abcdef)_{72}\,(gh)\}$ dim,	6
96	1	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta)(\eta\theta)$,	
	2	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ cyc,	
	3	$(abcd)$ all $(efgh)_4$,	8_1
	4	$(abcdef)_{48}\,(gh)$,	"

Order.	No.	Notation.		
	5	$(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta\eta\theta)_8$,		
	6	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)_8$ com$\}$ dim,		
	7	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)_8$ cyc$\}$ dim,		
	8	$\{(abcd)$ all $(efgh)_8$ pos$\}$ dim,		8_6
	9	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ all$\}_{4,4}$,		
	10	$\{(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta\eta\theta)$ pos$\}_{4,4}(\alpha\varepsilon\,.\,\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,		
	11	$\}(abcd)$ pos $(efgh)$ pos$\}_{4,4}(ae\,.\,bg\,.\,cf\,.\,dh)$,		8_9
	12	$\{(\alpha\beta)(\gamma\delta)(\varepsilon\zeta)(\eta\theta)\}$ pos $(\alpha\gamma\theta\,.\,\beta\delta\eta)(\alpha\delta\varepsilon\,.\,\beta\gamma\zeta)$,		
120	1	$(\alpha\beta\gamma\delta\varepsilon)_{20}(\zeta\eta\theta)$ all,		
	2	$(abcdef)_{60}(gh)$,	ins.	7_1
	3	$\{(abcdef)_{120}(gh)\}$ pos,	ins.	5
128	1	$(\alpha\beta\gamma\delta)_8(\varepsilon\zeta\eta\theta)_8(\alpha\varepsilon\,.\,\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,		
144	1	$(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta\eta\theta)$ pos,		
	2	$(\alpha\beta\gamma\delta\varepsilon\zeta)_{72}(\eta\theta)$,		
168	1	$(abcdefgh)_{168}$,	ins.	7
	2	$A\,(\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta)_8(\beta\gamma\varepsilon\delta\eta\theta\zeta)$,		
180	1	$(\alpha\beta\gamma\delta\varepsilon)$ pos $(\zeta\eta\theta)$ cyc,	ins.	
192	1	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)_8$,		
	2	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ all$\}_{4,4}(\alpha\varepsilon\,.\,\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,		
	3	$(\alpha\beta)(\gamma\delta)(\varepsilon\zeta)(\eta\theta)(\alpha\gamma\varepsilon\,.\,\beta\delta\zeta)(\gamma\varepsilon\eta\,.\,\delta\zeta\theta)$,		
	4	$+\,\{(\alpha\beta)(\gamma\delta)(\varepsilon\zeta)(\eta\theta)\}$ pos $(\alpha\gamma\varepsilon\,.\,\beta\delta\zeta)(\varepsilon\eta\,.\,\zeta\theta)$,		
	5	$\pm\,\{(\alpha\beta)(\gamma\delta)(\varepsilon\zeta)(\eta\theta)\}$ pos $(\alpha\gamma\varepsilon\,.\,\beta\delta\zeta)(\alpha\eta\beta\theta)$.		
240	1	$(abdef)_{120}(gh)$,	ins.	7
288	1	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ pos,		
	2	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ all$\}$ pos,		
	3	$(\alpha\beta\gamma\delta)$ pos $(\varepsilon\zeta\eta\theta)$ pos $(\alpha\varepsilon\,.\,\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,		
336	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta)_{336}$,	ins.	
360	1	$(\alpha\beta\gamma\delta\varepsilon)$ all $\zeta\eta\theta)$ cyc,	ins.	
	2	$(\alpha\beta\gamma\delta\varepsilon)$ pos $(\zeta\eta\theta)$ all,	ins.	
	3	$\{(\alpha\beta\gamma\delta\varepsilon)$ all $(\zeta\eta\theta)$ all$\}$ pos, ins.		
384	1	$(\alpha\beta)(\gamma\delta)(\varepsilon\zeta)(\eta\theta)(\alpha\gamma\varepsilon\,.\,\beta\delta\zeta)(\gamma\eta\,.\,\delta\theta)$,		
576	1	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ all,		
	2	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ all$\}$ pos $(\alpha\varepsilon\,.\,\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,		
	3	$\{(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ all$\}$ pos $(\alpha\varepsilon\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,		

Order.	No.	Notation.		
720	1	$\{(\alpha\beta\gamma\delta\varepsilon)$ all $(\zeta\eta\theta)$ all,	ins.	
	2	$(\alpha\beta\gamma\delta\varepsilon\zeta)$ pos $(\eta\theta)$*	ins.	
	3	$\{(abcdef)$ all $(gh)\}$ pos,	ins.	6
1152	1	$(\alpha\beta\gamma\delta)$ all $(\varepsilon\zeta\eta\theta)$ all $(\alpha\varepsilon\,.\,\beta\zeta\,.\,\gamma\eta\,.\,\delta\theta)$,		
1344	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta)_{1344}$,	ins.	
1440	1	$(\alpha\beta\gamma\delta\varepsilon\zeta)$ all $(\eta\theta)$,	ins.	
20160	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta)$ pos,	ins.	
40320	1	$(\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta)$ all,	ins.	
Total,	200			

EXPLANATIONS.

Degree Three.

(abc) all $\equiv 1,\ abc,\ acb,\ ab,\ ac,\ bc$.

(abc) cyc $\equiv 1,\ abc,\ acb$.

*Degree Four.**

$(ac\,.\,bd) \equiv 1,\ ac\,.\,bd$.

$(ab)(cd) \equiv 1,\ ab\,.\,cd,\ ab,\ cd$.

$(abcd)_4 \equiv 1,\ ab\,.\,cd,\ ac\,.\,bd,\ ad\,.\,bc$.

$(abcd)$ cyc $\equiv 1,\ ac\,.\,bd,\ abcd,\ adcb$.

$(abcd)_8 \equiv 1,\ ac\,.\,bd,\ ac,\ bd,\ ab\,.\,cd,\ ad\,.\,bc,\ abcd,\ adcb$.

$(abcd)$ pos $\equiv 1,\ ab\,.\,cd,\ ac\,.\,bd,\ ad\,.\,bc,\ abc,\ acd,\ bdc,\ adb,\ acb,\ bcd,\ abd,\ adc$.

$(abcd)$ all $\equiv (abcd)$ pos $+\ ab,\ cd,\ acbd,\ adbc,\ bc,\ ad,\ acdb,\ abdc,\ ac,\ bd,\ abcd,\ adcb$.

Degree Five.

$(abcde)$ cyc $\equiv 1,\ abcde,\ acebd,\ adbec,\ aedcb$.

(abc) cyc $(de) \equiv 1,\ de,\ abc,\ abc\,.\,de,\ acb,\ acb\,.\,de$.

$\{(abc)$ all $(de)\}$ pos $\equiv 1,\ abc,\ acb,\ ab\,.\,de,\ ac\,.\,de;\ bc\,.\,de$; i. e. the positive substitutions in the direct product of (abc) all and (de).

$(abcde)_{10} \equiv 1;\ abcde,\ acebd,\ adbec,\ aedcb,\ be\,.\,cd,\ ae\,.\,bd,\ ad\,.\,bc,\ ac\,.\,de,\ ab\,.\,ce$.

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*J. A. Serret, Lionville's Journal, 1850, pp. 52, 53.

(*abc*) all (*de*) $\equiv$ 1, *abc*, *acb*, *ab*, *ac*, *bc*, *de*, *abc* . *de*, *acb* . *de*, *ab* . *de*, *ac* . *de*, *bc* . *de*.

$(abcde)_{20}$* $\equiv$ 1, *abcde*, *acebd*, *adbec*, *aedcb*, *be* . *cd*, *ae* . *bd*, *ad* . *bc*, *ac* . *de*, *ab* . *ce*, *bced*, *acbe*, *aecd*, *abdc*, *adeb*, *bdec*, *adce*, *abed*, *aebc*, *acdb*.

Degree Six.

$\{(abcd)_4 (ef)\}$ dim $\equiv$ 1, *ab* . *cd*, *ac* . *bd* . *ef*, *ad* . *bc* . *ef* or the conjugates obtained by transforming this by (*abc*) cyc.

(*abc* . *def*) all $\equiv$ 1, *abc* . *def*, *acb* . *dfe*, *ab* . *de*, *ac* . *df*, *bc* . *ef*.

$(abcdef)_6 \equiv$ 1, *abc* . *def*, *acb* . *dfe*, *ad* . *bf* . *ce*, *af* . *be* . *cd*, *ae* . *bd* . *cf*.

$\{(abcd)_8$ com $(ef)\}$ dim $\equiv$ 1, *ac* . *bd*, *ac*, *bd*, *ab* . *cd* . *ef*, *ad* . *bc* . *ef*, *abcd* . *ef*, *adcb* . *ef*.

$\{(abcd)_8$ cyc $(ef)\}$ dim $\equiv$ 1, *ac* . *bd*, *abcd*, *adcb*, *ac* . *ef*, *bd* . *ef*, *ab* . *cd* . *ef*, *ad* . *bc* . *ef*.

$\{(abcd)_8$ pos $(ef)\}$ dim $\equiv$ 1, *ac* . *bd*, *ab* . *cd*, *ad* . *bc*, *ac* . *ef*, *bd* . *ef*, *abcd* . *ef*, *adcb* . *ef*.

$(abcdef)_{12} \equiv$ 1, *abc* . *def*, *acb* . *dfe*, *ab* . *de*, *ac* . *df*, *bc* . *ef*, *ad* . *be* . *cf*, *aecdbf*, *afbdce*, *ae* . *bd* . *cf*, *af* . *be* . *cd*, *ad* . *bf* . *ce*.

$(abcdef)_{12_2} \equiv$ 1, *ac* . *bd*, *ac* . *ef*, *bd* . *ef*, *abe* . *cdf*, *adf* . *bec*, *abf* . *cde*, *ade* . *bfc*, *aeb* . *cfd*, *afd* . *bce*, *afb* . *ced*, *aed* . *bcf*.

All the substitutions of $(abcdef)_{24_5}$ and $(abcdef)_{36_2}$ are given by Cole, Bulletin of the New York Mathematical Society, vol. II (1893), p. 185; also in Quarterly Journal of Mathematics, vol. XXVI, p. 372. A complete list of the substitutions in the remaining transitive groups of this degree with the exception of the alternating and the symmetric group is given by Cayley, Quarterly Journal of Mathematics, vol. XXV (1891), pp. 80–85. These groups are also given by Veronese, Annali di Matematica, vol. XI (1883), pp. 176–190.

Degree Seven. Cayley's list (Quarterly Journal, vol. XXV) contains all these groups with the exception of $\{(abcd)$ all (efg) all$\}_{4,1}$ and $(abcdefg)_{168}$. These two groups are given by Cole, Quarterly Journal of Mathematics, vol. XXVI, p. 373. The latter of these two groups is simple and it has received a great deal of attention. Cf. Kirkman, Proceedings of the Literary and Philosophical Society of Manchester, 1863, p. 65; also Moore, Bulletin of the American Mathematical Society, Vol. I (1894), p. 61.

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* Lagrange, Oeuvres, vol. III, p. 389.

Degree Eight. The given list by Cayley omits 43 of these groups. Cole's supplementary list (Bulletin of the New York Mathematical Society, vol. II, p. 184) omits two groups, $(abcd\,.\,efgh)$ pos $(af\,.\,be\,.\,cg\,.\,dh)$ and $(abcdefgh)_{1344}$, and it gives a notation for an intransitive group of order 16 which does not exist, Miller, Bulletin of the New York Mathematical Society, vol. III (1894), p. 168. The given group of order 1344 was studied by Jordan, Comptes rendus, vol. LXXIII; Noether, Mathematische Annalen, vol. XV, p. 90; Kirkman, Proceedings of the Literary and Philosophical Society of Manchester, vol. III, p. 150, etc. It is the only compound perfect group of degree 8, and there is no such group of any lower degree, Miller, American Journal of Mathematics, vol. XX, p. 280. Hence, there are six abstract perfect groups which may be represented as substitution groups whose degree does not exceed 8, viz. one group of each of the orders 60, 168, 360, 1344, 2520, and 20,160. $(abcdefgh)_{1344}$ is the holomorph of $A(abcdefgh)_8$. The holomorphs of $B(abcdefgh)_8$, $C(abcdefgh)_8$, $D(abcdefgh)_8$, and $(abcdefgh)$ cyc are $A\{(abcd)_8(efgh)_8\}_{2,2}$ $(ae\,.\,bf\,.\,cg\,.\,dh)$, $M\{(abcd)_8(efgh)_8\}$ dim $(ae\,.\,bf\,.\,cg\,.\,dh)$, $\ddagger$ $\{(ab)(cd)(ef)(gh)\}$ pos $(ace\,.\,bdf)(agbh)$, and $A\{(abcd)_8(efgh)_8\}_{2,2}(ae\,.\,bf\,.\,cg\,.\,dh)$ respectively. It is interesting to observe that the first and last of these four groups have the same holomorph.

Cornell University, *March*, 1899.

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On a Class of Equations of Transformation.

By Jacob Westlund.

The object of the following paper is to discuss those equations of transformation whose roots are the $n+1$ values of

$$y_{\frac{\mu}{\nu}} = \prod_{1,\,m}^{p} \mathrm{sn}^{2\alpha}.\,\mathrm{cn}^{\beta}.\,\mathrm{dn}^{\gamma}\,(4p\varpi/k),$$

where α, β, γ are any positive or negative integers and $\varpi = \frac{4\mu K + 4\nu i K'}{n}$, μ and ν being integers.

We use the notation $y_{\frac{\mu}{\nu}}$, since the value of the roots depends on the *ratio* of the integers μ and ν.[*] We also suppose, for simplicity, n to be an odd prime and set $n = 2m+1$.

The equation in question being rational for the domain of rationality, $R(x)$, where $x = k^2$, may be written

$$f_0(x)\,y^{n+1} + f_1(x)\,y^n + \ldots\ldots + f_{n+1}(x) = 0,$$

where

$$f_\alpha(x) = a_0 + a_1 x + \ldots\ldots + a_{m_\alpha} x^{m_\alpha} \tag{1}$$

The method for computing the coefficients $f_0 \ldots\ldots f_{n+1}$ is similar to that applied by Pierpont[†] to the modular equation, and consists of the following three steps:

1. To express the roots as q-series.
2. To find a superior limit of the degree in x of the equation.

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[*] Weber, "Elliptische Functionen," §67.

[†] "On Modular Equations" (Bulletin Am. Math. Soc., vol. III, No. 8, 1897).

3. To express x as a q-series and substitute for y one of the roots expressed as a q-series in the equation and equate the coefficients of the different powers of q to zero, by which we obtain a system of linear equations for determining the constants that enter into the coefficients $f_0 \ldots . f_{n+1}$.

1.—*To Develop the Roots into q-series.*

If we set

$$u_{\frac{\mu}{\nu}} = \prod_{1,\, m}^{p} \mathrm{sn}^2\,(4p\varpi),$$

$$v_{\frac{\mu}{\nu}} = \prod_{1,\, m}^{p} \mathrm{cn}\,(4p\varpi),$$

$$w_{\frac{\mu}{\nu}} = \prod_{1,\, m}^{p} \mathrm{dn}\,(4p\varpi),$$

we have

$$y_{\frac{\mu}{\nu}} = u^{\alpha}_{\frac{\mu}{\nu}} \,.\, v^{\beta}_{\frac{\mu}{\nu}} \,.\, w^{\gamma}_{\frac{\mu}{\nu}} \,.$$

I.—$u_{\frac{\mu}{\nu}}$:

Using the formula*

$$\prod_{1,\, m}^{p} \frac{\vartheta_1(v - p\omega)\,\vartheta_1(v + p\omega)}{\vartheta_0(v - p\omega)\,\vartheta_0(v + p\omega)} = (-1)^{\frac{n_1-1}{2}} \,.\, \frac{\vartheta_1}{\vartheta_0}\,(v_{\frac{\mu}{\nu}}/\tau_{\frac{\mu}{\nu}})\,\frac{\vartheta_0}{\vartheta_1}\,(v/\tau), \tag{2}$$

where

$$\tau_{\frac{\mu}{\nu}} = \frac{\tau + 16a}{n}, \quad v_{\frac{\mu}{\nu}} = v \;, \quad n_1 = 1, \quad \text{if } \nu \neq 0,$$

$$\tau_{\frac{\mu}{\nu}} = n\tau \qquad , \quad v_{\frac{\mu}{\nu}} = nv, \quad n_1 = n, \quad \text{if } \nu = 0,$$

and

$$\omega = \frac{\mu + \nu\tau}{n},$$

and observing that

$$\lim_{v=0} \frac{\vartheta_1(v_{\frac{\mu}{\nu}}/\tau_{\frac{\mu}{\nu}})}{\vartheta_1(v/\tau)} = n_1 \frac{\vartheta_0(0/\tau_{\frac{\mu}{\nu}})\,\vartheta_2(0/\tau_{\frac{\mu}{\nu}})\,\vartheta_3(0/\tau_{\frac{\mu}{\nu}})}{\vartheta_0(0/\tau)\,\vartheta_2(0/\tau)\,\vartheta_3(0/\tau)}$$

we get

$$(-1)^m \prod_{1,\, m}^{p} \frac{\vartheta_1^2}{\vartheta_0^2}\,(p\omega) = n_1(-1)^{\frac{n_1-1}{2}} \,.\, \frac{\vartheta_2(0/\tau_{\frac{\mu}{\nu}}) \,.\, \vartheta_3(0/\tau_{\frac{\mu}{\nu}})}{\vartheta_2(0/\tau) \,.\, \vartheta_3(0/\tau)} \tag{3}$$

---

* Königsberger, "Vorlesungen über Ellipt. Funct.," II, pp. 96–97.

and hence

$$\left.\begin{aligned} u_\infty &= \frac{n}{k^m}\cdot\frac{\vartheta_2(0/n\tau)\,.\,\vartheta_3(0/n\tau)}{\vartheta_2(0/\tau)\,.\,\vartheta_3(0/\tau)}, \\ \underset{a=0,\,1,\,\ldots,\,n-1}{u_a} &= \frac{(-1)^m}{k^m}\cdot\frac{\vartheta_2\left(0\Big/\frac{\tau+16a}{n}\right)\vartheta_3\left(0\Big/\frac{\tau+16a}{n}\right)}{\vartheta_2(0/\tau)\,.\,\vartheta_3(0/\tau)}, \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} u_\infty &= n\,.\,2^{-2m}(1+a_1q+\ldots\ldots), \\ \underset{a=0,\,1,\,\ldots,\,n-1}{u_a} &= (-1)^m.\,2^{-2m}.\,q^{-\frac{2r}{n}}.\,\varepsilon^{2a}(1+b_1\,\varepsilon^{8a}q^{\frac{1}{n}}+\ldots\ldots), \end{aligned}\right\} \tag{5}$$

where $r=\frac{n^2-1}{8}$ and $\varepsilon=e^{\frac{2\pi i}{n}}$.

II.—$v_{\frac{\mu}{\nu}}$.

We have*

$$\prod_{1,\,m}^{p}\frac{\vartheta_3(v-p\omega)\,\vartheta_2(v+p\omega)}{\vartheta_0(v-p\omega)\,\vartheta_0(v+p\omega)}=\frac{\vartheta_2}{\vartheta_0}(v_{\frac{\mu}{\nu}}/\tau_{\frac{\mu}{\nu}})\,.\,\frac{\vartheta_0}{\vartheta_2}(v/\tau). \tag{6}$$

Hence
$$\prod_{1,\,m}^{p}\frac{\vartheta_2^2}{\vartheta_0^2}(p\omega)=\frac{\vartheta_2}{\vartheta_0}(0/\tau_{\frac{\mu}{\nu}})\,.\,\frac{\vartheta_0}{\vartheta_2}(0/\tau)$$

and
$$\begin{aligned} v_{\frac{\mu}{\nu}}^2 &= \left(\sqrt{\frac{k'}{k}}\right)^n.\,\frac{\vartheta_2}{\vartheta_0}(0/\tau_{\frac{\mu}{\nu}}) \\ &= \frac{\psi^{2n}(\tau)}{\phi^{2n}(\tau)}\cdot\frac{\phi^2(\tau_{\frac{\mu}{\nu}})}{\psi^2(\tau_{\frac{\mu}{\nu}})}, \end{aligned}$$

where $\phi(\tau)=\sqrt[4]{k}$; $\psi(\tau)=\sqrt[4]{k'}$.

Hence
$$v_{\frac{\mu}{\nu}}=\rho_{\frac{\mu}{\nu}}\cdot\frac{\psi^n(\tau)}{\phi^n(\tau)}\cdot\frac{\phi(\tau_{\frac{\mu}{\nu}})}{\psi(\tau_{\frac{\mu}{\nu}})}, \tag{7}$$

where $\rho_{\frac{\mu}{\nu}}=\pm 1$.

In order to determine ρ_∞ we suppose $\tau=r+is$, and let s become infinite. Then q approaches zero and we get

$$\rho_\infty=(-1)^r.$$

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* Königsberger, loc. cit.

To determine ρ_0 we give to τ the value $i\infty$, and we find* that

$$\rho_0 = (-1)^r.$$

In order to determine $\underset{a=1,2,\ldots,n-1}{\rho_a}$, we proceed as follows: Let

$$f(v, \varkappa) = 0$$

be the equation whose roots are

$$v_{\substack{\mu \\ \nu}} = \prod_{1,m}^{p} \operatorname{cn}(4p\varpi/k);$$

then

$$f\left\{(-1)^r \frac{\psi^n(\tau)}{\phi^n(\tau)} \cdot \frac{\phi\left(\dfrac{\tau}{n}\right)}{\psi\left(\dfrac{\tau}{n}\right)},\ \varkappa(\tau)\right\} = 0.$$

In this identity replace τ by $\tau + 16a$ and we get

$$f\left\{(-1)^r \frac{\psi^n(\tau)}{\phi^n(\tau)} \cdot \frac{\phi\left(\dfrac{\tau+16a}{n}\right)}{\psi\left(\dfrac{\tau+16a}{n}\right)}, \varkappa(\tau)\right\} = 0.$$

which shows that

$$\rho_a = (-1)^r. \qquad a = 0, 1, \ldots, n-1)$$

Hence

$$\left.\begin{aligned} v_\infty &= (-1)^r \frac{\psi^n(\tau)}{\phi^n(\tau)} \cdot \frac{\phi(n\tau)}{\psi(n\tau)}, \\ \underset{a=0,1,\ldots,n-1}{v_a} &= (-1)^r \frac{\psi^n(\tau)}{\phi^n(\tau)} \cdot \frac{\phi\left(\dfrac{\tau+16a}{n}\right)}{\psi\left(\dfrac{\tau+16a}{n}\right)}, \end{aligned}\right\} \qquad (8)$$

or

$$\left.\begin{aligned} v_\infty &= (-1)^r . 2^{-m}(1 + a_1 q + \ldots.), \\ \underset{a=0,1,\ldots,n-1}{v_a} &= (-1)^r 2^{-m} . q^{-\frac{r}{n}} \varepsilon^a (1 + \varepsilon^{8a} b_1 q^{\frac{1}{n}} + \ldots.). \end{aligned}\right\} \qquad (9)$$

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* Weber, "Ellipt. Funct.," p. 65.

III.—$w_{\frac{\mu}{\nu}}$.

We have*

$$\prod_{1,\,m}^{p} \frac{\vartheta_3(v-p\omega)\,\vartheta_3(v+p\omega)}{\vartheta_0(v-p\omega)\,\vartheta_0(v+p\omega)} = \frac{\vartheta_3}{\vartheta_0}\left(v_{\frac{\mu}{\nu}}/\tau_{\frac{\mu}{\nu}}\right).\frac{\vartheta_0}{\vartheta_3}(v/\tau). \tag{10}$$

Hence

$$\prod_{1,\,m}^{p} \frac{\vartheta_3^2}{\vartheta_0^2}(p\omega) = \frac{\vartheta_3}{\vartheta_0}\left(0/\tau_{\frac{\mu}{\nu}}\right).\frac{\vartheta_0}{\vartheta_3}(0/\tau)$$

and

$$\begin{aligned} w^2_{\frac{\mu}{\nu}} &= (\sqrt{k'})^n.\frac{\vartheta_3}{\vartheta_0}\left(0/\tau_{\frac{\mu}{\nu}}\right) \\ &= \frac{\psi^{2n}(\tau)}{\psi^2\left(\tau_{\frac{\mu}{\nu}}\right)}. \end{aligned}$$

Hence

$$w_{\frac{\mu}{\nu}} = \rho_{\frac{\mu}{\nu}}\frac{\psi^n(\tau)}{\psi\left(\tau_{\frac{\mu}{\nu}}\right)}, \tag{11}$$

where $\rho_{\frac{\mu}{\nu}} = \pm 1$.

Reasoning as before, we get

$$\rho_\infty = 1, \quad \rho_a = (-1)^r. \qquad a = 0, 1, \ldots\ldots, n-1$$

Hence

$$\left.\begin{aligned} w_\infty &= \frac{\psi^n(\tau)}{\psi(n\tau)}, \\ \underset{a=0,1,\ldots\ldots,n-1}{w_a} &= (-1)^r\frac{\psi^n(\tau)}{\psi\left(\frac{\tau+16a}{n}\right)}, \end{aligned}\right\} \tag{12}$$

or

$$\left.\begin{aligned} w_\infty &= 1 + a_1 q + \ldots\ldots \\ \underset{a=0,1,\ldots\ldots,n-1}{w_a} &= (-1)^r\left\{1 + \varepsilon^{8a} b_1 q^{\frac{1}{n}} + \ldots\ldots\right\}. \end{aligned}\right. \tag{13}$$

Having thus obtained expressions for the three quantities $u_{\frac{\mu}{\nu}}$, $v_{\frac{\mu}{\nu}}$, $w_{\frac{\mu}{\nu}}$, we get the following expressions for $y_{\frac{\mu}{\nu}}$:

$$\left.\begin{aligned} y_\infty &= (-1)^{\beta r}.n^a.\frac{\phi^{2a+\beta}(n\tau)\,.\,\psi^{n(\beta+\gamma)}(\tau)\,.\,\vartheta_3^{2a}(0/n\tau)}{\phi^{n(2a+\beta)}(\tau)\,.\,\psi^{\beta+\gamma}(n\tau)\,.\,\vartheta_3^{2a}(0/\tau)}, \\ \underset{a=0,1,\ldots\ldots,n-1}{y_a} &= (-1)^{ma+r(\beta+\gamma)}.\frac{\phi^{2a+\beta}\left(\frac{\tau+16a}{n}\right).\,\psi^{n(\beta+\gamma)}(\tau).\,\vartheta_3^{2a}\left(0\Big/\frac{\tau+16a}{n}\right)}{\phi^{n(2a+\beta)}(\tau).\,\psi^{\beta+\gamma}\left(\frac{\tau+16a}{n}\right).\,\vartheta_3^{2a}(0/\tau)}, \end{aligned}\right\} \tag{14}$$

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* Königsberger, loc. cit.

or

$$\left.\begin{aligned} y_\infty &= (-1)^{\beta r}.\, n^{\alpha}.\, 2^{-m(2\alpha+\beta)}(1 + a_1 q + \ldots.), \\ \underset{a=0,1,\ldots,n-1}{y_a} &= (-1)^{ma+r(\beta+\gamma)}.\, 2^{-m(2\alpha+\beta)}.\, \varepsilon^{a(2\alpha+\beta)}.\, q^{\frac{-r(2\alpha+\beta)}{n}}(1 + b_1\varepsilon^{8a}q^{\frac{1}{n}} + \ldots. \end{aligned}\right\} \quad (15)$$

2.—*A Superior Limit of the Degree of x.*

To determine this we have to determine the sum of the orders of the infinities of the roots for the infinite x-plane. The only points in the x-plane for which the roots may vanish or become infinite are $x = 0, 1, \infty$. We will, therefore, determine the value of the roots for these three values of x.

I. $x = 0$. From (15) we get

$$\left.\begin{aligned} (y_\infty)_{x=0} &= (-1)^{\beta r}.\, n^{\alpha}.\, 2^{-m(2\alpha+\beta)}, \\ (y_a)_{x=0} &= \infty \text{ of order } \frac{r(2\alpha+\beta)}{n}. \end{aligned}\right\} \quad (16)$$

II. $x = 1$. It is easy to show, on passing to exponentials, that

$$\left.\begin{aligned} (u_0)_{x=1} &= (-1)^m.\, n, \\ (u_{\infty,1,\ldots,n-1})_{x=1} &= 1, \end{aligned}\right\} \quad (17)$$

For $v_{\frac{\mu}{\nu}}$ we have

$$\begin{aligned} v_\infty(\tau) &= \prod_{1,m}^{p} \frac{1}{\operatorname{cn}\left(\frac{4piL'}{n}\Big| k'\right)} \\ &= \frac{1}{v_0(\tau_1)} \qquad\qquad \text{where } \tau_1 = -\frac{1}{\tau} \\ &= (-1)^r.\, 2^m q_1^{\frac{r}{n}}(1 + a_1 q_1^{\frac{1}{n}} + \ldots.). \qquad q_1 = e^{\pi i\tau_1}. \end{aligned} \quad (18)$$

Similarly we get

$$\begin{aligned} v_0(\tau) &= \frac{1}{v_\infty(\tau_1)} \\ &= (-1)^r\, 2^m(1 + b_1 q_1 + \ldots.) \end{aligned} \quad (19)$$

and

$$\begin{aligned} \underset{a=1\ldots n-1}{v_a(\tau)} &= \frac{1}{v_b(\tau_1)} \\ &= (-1)^r.\, 2^m q_1^{\frac{r}{n}}\varepsilon^{-b}(1 + c_1\varepsilon^{8b}q_1^{\frac{1}{n}} + \ldots.). \end{aligned} \quad (20)$$

Hence

$$\left.\begin{aligned} (v_0)_{\kappa=1} &= (-1)^r . 2^m, \\ (v_{\infty, 1 \dots n})_{\kappa=1} &= 0 \text{ of order } \frac{r}{n} . \end{aligned}\right\} \tag{21}$$

For $w_{\frac{\mu}{\nu}}$ we obtain by a similar reasoning

$$\left.\begin{aligned} w_\infty(\tau) &= \frac{w_0(\tau_1)}{v_0(\tau_1)} = 2^m . q_1^{\frac{r}{n}} (1 + a_1 q_1^{\frac{1}{n}} + \dots .), \\ w_0(\tau) &= \frac{w_\infty(\tau_1)}{v_\infty(\tau_1)} = (-1)^r . 2^m (1 + b_1 q_1 + \dots .), \\ \underset{a=1 \dots . n-1}{w_a(\tau_1)} &= \frac{w_b(\tau_1)}{v_b(\tau_1)} = 2^m . q_1^{\frac{r}{n}} . \varepsilon^{-b} (1 + c_1 \varepsilon^{8b} q^{\frac{1}{n}} + \dots .), \end{aligned}\right\} \tag{22}$$

or

$$\left.\begin{aligned} (w_0)_{\kappa=1} &= (-1)^r . 2^m, \\ (w_{\infty, 1 \dots n-1})_{\kappa=1} &= 0 \text{ of order } \frac{r}{n} . \end{aligned}\right\} \tag{23}$$

Hence from (17), (21) and (23),

$$\left.\begin{aligned} (y_0)_{\kappa=1} &= (-1)^{ma + r(\beta+\gamma)} . n^a . 2^{m(\beta+\gamma)}, \\ (y_{\infty, 1, \dots, n-1})_{\kappa=1} &= 0 \text{ of order } \frac{r}{n}(\beta+\gamma), \end{aligned}\right\} \tag{24}$$

III. $\kappa = \infty$

We have

$$u_\infty(\tau) = \lambda^{2m} \prod_{1, m}^{p} \operatorname{sn}^2 \left(4p \frac{L + iL'}{n} \Big/ \lambda \right) = u_a(\tau_1),$$

where $\lambda = \frac{1}{k}$, $\tau_1 = \frac{\tau}{1-\tau}$, $16a \equiv 1 \bmod n$.

Similarly,

$$u_a(\tau) = \lambda^{2m} u_\infty(\tau_1) \text{ if } 16a \equiv -1 \bmod n,$$
$$u_a(\tau) = \lambda^{2m} u_b(\tau_1) \text{ if } (16b-1)a \equiv -b \bmod n.$$

Hence

$$\left.\begin{aligned} u_\infty(\tau) &= (-1)^m . 2^{2m} q_1^{m - \frac{2r}{n}} . \varepsilon^{2a} (1 + a_1 \varepsilon^{8a} q_1^{\frac{1}{n}} + \dots .), \\ &\qquad \text{where } 16a \equiv 1 \bmod n, \\ u_a(\tau) &= n . 2^{2m} q_1^m (1 + b_1 q_1 + \dots .), \\ &\qquad \text{where } 16a \equiv -1 \bmod n, \\ u_a(\tau) &= (-1)^m . 2^{2m} q_1^{m - \frac{2r}{n}} . \varepsilon^{2b} (1 + c_1 \varepsilon^{8b} q_1^{\frac{1}{n}} + \dots .), \\ &\qquad \text{where } (16b-1)a \equiv -b \bmod n. \end{aligned}\right\} \tag{25}$$

In the same way we get

$$\left.\begin{array}{l} v_\infty(\tau) = w_a(\tau) = (-1)^r(1 + a_1\varepsilon^{8a}q_1^{\frac{1}{n}} + \ldots.), \\ \qquad\qquad \text{where } 16a \equiv 1 \bmod n, \\ v_a(\tau) = w_\infty(\tau_1) = 1 + b_1q_1 + \ldots.. \\ \qquad\qquad \text{where } 16a \equiv -1 \bmod n, \\ v_a(\tau) = w_b(\tau_1) = (-1)^r(1 + c_1\varepsilon^{8b}q_1^{\frac{1}{n}} + \ldots., \\ \qquad\qquad \text{where } (16b-1)\,a \equiv -b \bmod n. \end{array}\right\} \tag{26}$$

and

$$\left.\begin{array}{l} w_\infty(\tau) = v_a(\tau_1) = (-1)^r . 2^{-m}q_1^{-\frac{r}{n}} . \varepsilon^a(1 + a_1\varepsilon^{8a}q_1^{\frac{1}{n}} + \ldots.), \\ \qquad\qquad \text{where } 16a \equiv 1 \bmod n, \\ w_a(\tau) = v_\infty(\tau_1) = (-1)^r . 2^{-m}(1 + b_1q_1 + \ldots.), \\ \qquad\qquad \text{where } 16a \equiv -1 \bmod. n, \\ w_b(\tau) = v_b(\tau_1) = (-1)^r . 2^{-m}q_1^{-\frac{r}{n}} . \varepsilon^b(1 + c_1\varepsilon^{8b}q_1^{\frac{1}{n}} + \ldots.), \\ \qquad\qquad \text{where } (16b-1)\,a \equiv -b \bmod n. \end{array}\right\} \tag{27}$$

Hence

$$\left.\begin{array}{l} (y_a)_{\kappa=\infty} = 0 \text{ of order } \alpha m, \text{ if } 16a \equiv -1 \bmod n, \\ (y_{\infty,\,a})_{\kappa=\infty} = 0 \text{ of order } \alpha m - \dfrac{r}{n}(2\alpha+\gamma), \text{ if } 16a \not\equiv -1 \bmod n. \end{array}\right\} \tag{28}$$

3.—*Equations that can be derived from a given Equation of Transformation by Means of Linear Transformations.*

By applying a linear transformation to the equation whose roots are y_μ we obtain other equations of transformation. The results we obtain are given in the table below, where to the left the substitution to be made and to the right the roots of the resulting equation are given. κ' is defined by $\kappa + \kappa' = 1$.

1. $\begin{pmatrix} \kappa\,, & y \\ -\dfrac{\kappa}{\kappa'}\,; & \kappa'^{\alpha m}y \end{pmatrix}, \quad \prod\limits_{1,\,m}^{p} \text{sn}^{2\alpha} . \text{cn}^{\beta} . \text{dn}^{-(2\alpha+\beta+\gamma)}(4p\varpi/k).$

2. $\begin{pmatrix} \kappa\,, & y \\ \kappa'\,, & (-1)^{\alpha m}y \end{pmatrix}, \quad \prod\limits_{1,\,m}^{p} \text{sn}^{2\alpha} . \text{cn}^{-(2\alpha+\beta+\gamma)} . \text{dn}^{\gamma}(4p\varpi/k).$

$$3.\quad \begin{pmatrix} x\ , & y \\ -\dfrac{x'}{x}\,, & (-1)^{am}.\,x^{am}.\,y \end{pmatrix},\quad \prod_{1,\,m}^{p} \mathrm{sn}^{2a}.\,\mathrm{cn}^{\gamma}.\,\mathrm{dn}^{-(2a+\beta+\gamma)}\,(4p\varpi/k).$$

$$4.\quad \begin{pmatrix} x\ , & y \\ \dfrac{1}{x'}\,, & (-1)^{am}.\,x'^{am}.\,y \end{pmatrix},\quad \prod_{1,\,m}^{p} \mathrm{sn}^{2a}.\,\mathrm{cn}^{-(2a+\beta+\gamma)}.\,\mathrm{dn}^{\beta}\,(4p\varpi/k).$$

$$5.\quad \begin{pmatrix} x\ ; & y \\ \dfrac{1}{x}\,, & x^{am}.\,y \end{pmatrix},\quad \prod_{1,\,m}^{p} \mathrm{sn}^{2a}.\,\mathrm{cn}^{\gamma}.\,\mathrm{dn}^{\beta}\,(4p\varpi/k).$$

Hence, using the notation

$$y_{\frac{\mu}{\nu},\,a,\,\beta,\,\gamma} = \prod_{1,\,m}^{p} \mathrm{sn}^{2a}.\,\mathrm{cn}^{\beta}.\,\mathrm{dn}^{\gamma}\,(4p\varpi/k),$$

the substitutions given above lead to equations whose roots are

$$1.\quad y_{\frac{\mu}{\nu},\,a,\,\beta,\,-(2a+\beta+\gamma)},$$

$$2.\quad y_{\frac{\mu}{\nu},\,a,\,-(2a+\beta+\gamma),\,\gamma},$$

$$3.\quad y_{\frac{\mu}{\nu},\,a,\,\gamma,\,-(2a+\beta+\gamma)},$$

$$4.\quad y_{\frac{\mu}{\nu},\,a,\,-(2a+\beta+\gamma),\,\beta},$$

$$5.\quad y_{\frac{\mu}{\nu},\,a,\,\gamma,\,\beta}$$

respectively.

4.—*Applications.*

I. *The equation whose roots are*

$$y_{\frac{\mu}{\nu}} = \prod_{1,\,m}^{p} \frac{\mathrm{dn}^2}{\mathrm{cn}}\,(4p\varpi/k).$$

From (15) we get

$$\left.\begin{aligned} y_{\infty} &= (-1)^{r}.\,2^{m}\,(1+\ldots\ldots), \\ \underset{a=0,\,1,\,\ldots\ldots,\,n-1}{y_{a}} &= (-1)^{r}.\,2^{m}.\,\varepsilon^{-ar}.\,q^{\frac{r}{n}}(1+\ldots\ldots), \end{aligned}\right\} \qquad (29)$$

and from (29) we get the values of the roots for $x=0, 1, \infty$ as follows:

$$\left.\begin{array}{l} x=0\left\{\begin{array}{ll} y_\infty & =(-1)^r.\,2^m, \\ \underset{a=0,\,1,\,\ldots,\,n-1}{y_a} & =0 \text{ of order } \dfrac{r}{n}, \end{array}\right. \\ x=1\left\{\begin{array}{ll} y_0 & =(-1)^r.\,2^m, \\ \underset{a=1,\,\ldots,\,n-1}{y_{\infty,\,a}} & =0 \text{ of order } \dfrac{r}{n}, \end{array}\right. \\ x=\infty\left\{\begin{array}{ll} y_a & =2^{-2m}, \text{ if } 16a\equiv -1 \bmod n, \\ y_{\infty,\,a} & =\infty \text{ of order } \dfrac{2r}{n} \text{ if } 16a \not\equiv -1 \bmod n. \end{array}\right. \end{array}\right\} \tag{30}$$

Since the equation remains unchanged if we replace x by x', it must be of the form

$$y^{n+1}+f_1(xx')\,y^n+\ldots\ldots+f_{n+1}(xx')=0, \tag{31}$$

where

$$f_s(xx')=b_{s,\,0}+b_{s,\,1}xx'+\ldots\ldots+b_{s,\,m_s}(xx')^{m_s} \text{ and } m_s\leq r.$$

The term $f_{n+1}(xx')$ being equal to the product of the roots, can easily be determined, and we get

$$f_{n+1}(xx')=(xx')^r. \tag{32}$$

For $x=0$, n roots vanish. Hence $f_2, f_3, \ldots\ldots, f_n$ must all contain the factor xx' and also

$$b_{1,\,0}=(-1)^{r+1}.\,2^m. \tag{33}$$

For $x=\infty$, n roots become infinite. Hence we see that

$$\begin{array}{l} m_s<r, \quad s=1,\,2,\,\ldots\ldots,\,n-1, \\ m_n=r, \end{array} \quad \text{and} \quad b_{n,\,m_n}=-2^{2m}. \tag{34}$$

For $n=3$ the equation is

$$y^4+2y^3-4xx'y+xx'=0. \tag{35}$$

From this equation we derive two other equations by the substitutions

$$\begin{pmatrix} x, & y \\ -\dfrac{x}{x'}, & y \end{pmatrix} \text{ and } \begin{pmatrix} x, & y \\ -\dfrac{x'}{x}, & y \end{pmatrix},$$

as shown in the table below:

ROOTS.	EQUATIONS.
$\prod\limits_{1,\,m}^{p} \frac{\mathrm{dn}^2}{\mathrm{cn}}(4p\varpi/k)$	$y^4 + 2y^3 - 4\varkappa\varkappa' y + \varkappa\varkappa' = 0$
$\prod\limits_{1,\,m}^{p} \frac{1}{\mathrm{cn}\,.\,\mathrm{dn}}(4p\varpi/k)$	$y^4 + 2y^3 + \frac{4\varkappa}{\varkappa'^2} y - \frac{\varkappa}{\varkappa'^2} = 0$
$\prod\limits_{1,\,m}^{p} \frac{\mathrm{cn}^2}{\mathrm{dn}}(4p\varpi/k)$	$y^4 + 2y^3 + \frac{4\varkappa'}{\varkappa^3} y - \frac{\varkappa'}{\varkappa^3} = 0$

II. *The equation whose roots are*

$$y_{\frac{\mu}{\nu}} = \prod_{1,\,m}^{p} \frac{\mathrm{cn}}{\mathrm{sn}^2}(4p\varpi/k).$$

From (15) we get

$$\left.\begin{aligned} y_\infty &= \frac{(-1)^r \,.\, 2^m}{n}(1 + \ldots\ldots), \\ &\vdots \\ \underset{a=0,\,1,\,\ldots\ldots,\,n-1}{y_a} &= (-1)^{r-m} \,.\, 2^m \,.\, \varepsilon^{-a} \,.\, q^{\frac{r}{n}}(1 + \ldots\ldots), \end{aligned}\right\} \qquad (36)$$

and from (29),

$$\left.\begin{aligned} &\varkappa = 0 \begin{cases} y_\infty = \dfrac{(-1)^r \,.\, 2^m}{n}, \\ \underset{a=0,\,1,\,\ldots\ldots,\,n-1}{y_a} = 0 \text{ of order } \dfrac{r}{n}, \end{cases} \\ &\varkappa = 1 \begin{cases} y_0 = \dfrac{(-1)^{r-m} \,.\, 2^m}{n}, \\ \underset{a=1,\,\ldots\ldots,\,n-1}{y_{\infty,\,a}} = 0 \text{ of order } \dfrac{r}{n}, \end{cases} \\ &\varkappa = \infty \begin{cases} y_a = \infty \text{ of order } m, & 16a \equiv -1 \bmod n \\ y_{\infty,\,a} = \infty \text{ of order } -\dfrac{2r}{n} + m, & 16a \not\equiv -1 \bmod n \end{cases} \end{aligned}\right\} \qquad (37)$$

If we replace $\varkappa$ by $\varkappa'$, the equation is transformed into an equation whose roots are

$$(-1)^m \prod_{1,\,m}^{p} \frac{\mathrm{cn}}{\mathrm{sn}^2}(4p\varpi/k).$$

Hence if m is even, the equation must be of the form

$$y^{n+1}+f_1(xx')\,y^n+\ldots\ldots+f_{n+1}(xx')=0, \tag{38}$$

and if m is odd,

$$y^{n+1}+(x-x')f_1(xx')\,y^n+f_2(xx')\,y^{n-1}+\ldots\ldots+f_{n+1}(xx')=0, \tag{39}$$

where

$$f_s(xx')=b_{s,\,0}+b_{s,\,1}xx'+\ldots\ldots+b_{s,\,m_s}(xx')^{m_s}.$$

When m is even, we have $m_s\leqq r$, and when m is odd, we have

$$m_s\leqq\begin{cases} r & \text{if } s \text{ is even,}\\ r-1 & \text{if } s \text{ is odd.}\end{cases}$$

For $f_{n+1}(xx')$ we get the expression

$$f_{n+1}(xx')=\frac{(-1)^m}{n}(xx')^r. \tag{40}$$

The coefficients $f_2,\ldots\ldots,f_n$ must all contain the factor xx', and we also get

$$b_{1,\,0}=\frac{(-1)^{r+1}\,.\,2^m}{n}. \tag{41}$$

From the above, and making use of the fact that for $x=\infty$ all the roots become infinite, we obtain the equation for $n=3$. This equation and those that can be derived from it by the substitutions

$$\begin{pmatrix} x\,, & y\\ -\dfrac{x}{x'}\,, & x'^{-m}y\end{pmatrix} \text{ and } \begin{pmatrix} x\,, & y\\ -\dfrac{x'}{x}\,, & (-1)^m x^{-m}y\end{pmatrix}$$

are given below.

ROOTS.	EQUATIONS.
$\prod\limits_{1,\,m}^{p}\dfrac{\text{cn}}{\text{sn}^2}(4p\varpi/k)$	$y^4-\frac{2}{3}(x-x')\,y^3-\dfrac{xx'}{3}=0$
$\prod\limits_{1,\,m}^{p}\dfrac{\text{cn}\,.\,\text{dn}}{\text{sn}^2}(4p\varpi/k)$	$y^4+\frac{2}{3}(1+x)\,y^3+\dfrac{xx'^2}{3}=0$
$\prod\limits_{1,\,m}^{p}\dfrac{\text{dn}}{\text{sn}^2}(4p\varpi/k)$	$y^4-\frac{2}{3}(1+x')\,y^3+\dfrac{x'x^2}{3}=0$

III. *The equation whose roots are*

$$y_{\frac{\mu}{\nu}} = \prod_{1,\,m}^{p} \mathrm{sn}^2 . \mathrm{cn} . \mathrm{dn}\,(4p\varpi/k).$$

From (15) we get

$$\left.\begin{aligned} y_\infty &= (-1)^r n . 2^{-3m}(1 + \dots .), \\ y_a &= (-1)^m . 2^{-3m} . \varepsilon^{8a} . q^{-\frac{3r}{n}}(1 + \dots .), \end{aligned}\right\} \qquad (42)$$

and from (29),

$$\left.\begin{aligned} &x = 0 \begin{cases} y_\infty & = (-1)^r . n . 2^{-3m}, \\ \underset{a=0,\,1,\,\dots,\,n-1}{y_a} & = \infty \text{ of order } \dfrac{3r}{n}, \end{cases} \\ &x = 1 \begin{cases} y_0 & = (-1)^m . n . 2^{2m}, \\ \underset{a=1,\,\dots,\,n-1}{y_{\infty,\,a}} & = 0 \text{ of order } \dfrac{2r}{n}, \end{cases} \\ &x = \infty \begin{cases} y_a & = 0 \text{ of order } m, & 16a \equiv -1 \bmod n \\ y_{\infty,\,a} & = 0 \text{ of order } \dfrac{mn - 3r}{n}. & 16a \not\equiv -1 \bmod n \end{cases} \end{aligned}\right\} \qquad (43)$$

The equation must be of the form

$$y^{n+1} + \frac{g_1(x)}{x^{r_1}} y^n + \dots . + \frac{g_n(x)}{x^{r_n}}\, y + \frac{A . x'^{\rho}}{x^{r_{n+1}}} = 0,$$

where $A =$ constant and $g_s(x) = a_{s,\,0} + a_{s,\,1}x + \dots . + a_{s,\,m_s}x^{m_s}$, and it is also seen that the degree of x cannot exceed $3r$.

The last term being the product of the roots, we get

$$r_{n+1} = 3r, \quad A = (-1)^{r+m} . n. \qquad (44)$$

We have also

$$\frac{g_n(x)}{x^{r_n}} = -y_\infty \cdot y_0 \dots . y_{n-1}\left(\frac{1}{y_\infty} + \dots . + \frac{1}{y_{n-1}}\right),$$

which gives

$$r_n = 3r, \quad a_{n,\,0} = (-1)^{m+1} . 2^{3m}. \qquad (45)$$

In order to determine r_1 and a superior limit of $r_2, r_3, \dots ., r_{n-1}$, we use the method employed by Pierpont.* Setting

$$S_p = y_\infty^p + y_0^p + \dots . + y_{n-1}^p,$$

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* "On Modular Equations."

and denoting the order of infinity of S_p by σ_p, we get

$$\sigma_p = \frac{3pr - \nu_p}{n}, \qquad p = 1 . 2, \ldots , n - 1 \tag{46}$$

where ν_p is the smallest positive root of

$$8\nu + 3p \equiv 0 \mod n,$$

and hence

$$r_1 = \sigma_1,$$

and if $\sigma_1, \sigma_2, \ldots$ form a regular sequence,* we also get

$$r_p \leqq \sigma_p . \qquad p = 2, 3, \ldots , n - 1 \tag{47}$$

In order to determine ρ, m_1, m_n and a superior limit of $m_2, m_3, \ldots , m_{n-1}$, we make use of the fact that the equation remains unchanged for the substitution

$$\begin{pmatrix} x , & y \\ \frac{1}{x} , & x^m y \end{pmatrix},$$

From this fact we get the relations

$$\rho = 2r, \quad m_p = 2r_p - pm . \qquad (p = 1, 2, \ldots , n), \; a_{\alpha, \lambda} = a_{\alpha, m_\alpha - \lambda} \tag{48}$$

For $x = 1$, n roots vanish and one root $= (-1)^m n . 2^{3m}$. Hence

$$\left.\begin{array}{l} a_{1,0} + a_{1,1} + \ldots + a_{1,m_1} = (-1)^{m+1} . n . 2^{2m}, \\ a_{2,0} + a_{2,1} + \ldots + a_{2,m_2} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n,0} + a_{n,1} + \ldots + a_{n,m_n} = 0 . \end{array}\right\} \tag{49}$$

The relations given above show that for $n = 3$ the equation is

$$y^4 + \frac{6(1 + x)}{x} y^3 + \frac{a_{2,0}(1 - 2x + x^2)}{x^2} y^2 + \frac{8(1 - x - x^2 + x^3)}{x^3} y + \frac{3x^2}{x^3} = 0 .$$

But for $x = \infty$ the equation reduces to

$$y(y + 2)^3 = 0 .$$

Hence

$$a_{2,0} = 12 .$$

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* Pierpont, l. c.

From this equation we derive two other equations whose roots are

$$\prod_{1,m}^{p} \frac{\text{sn}^2\,\text{dn}}{\text{cn}^4}(4p\varpi/k) \quad \text{and} \quad \prod_{1,m}^{p} \frac{\text{sn}^2\,\text{cn}}{\text{dn}^4}(4p\varpi/k)$$

by the substitutions

$$\begin{pmatrix} x & y \\ x' & (-1)^m y \end{pmatrix} \text{ and } \begin{pmatrix} x\ , & y \\ -\dfrac{x}{x'}\ , & x'^m y \end{pmatrix} \text{ resp.}$$

Thus we have for $n=3$:

ROOTS.	EQUATIONS.
$\prod_{1,m}^{p} \text{sn}^2 . \text{cn} . \text{dn}\,(4p\varpi/k)$	$y^4 + \frac{6(1+x)}{x} y^3 + \frac{12x'^2}{x^2} y^2 + \frac{8x'^2(1+x)}{x^3} y + \frac{3x'^2}{x^3} = 0$
$\prod_{1,m}^{p} \frac{\text{sn}^2\,\text{dn}}{\text{cn}^4}(4p\varpi/k)$	$y^4 - \frac{6(1+x')}{x'} y^3 + \frac{12x^2}{x'^2} y^2 - \frac{8x^2(1+x')}{x'^3} y + \frac{3x^2}{x'^3} = 0$
$\prod_{1,m}^{p} \frac{\text{sn}^2\,\text{cn}}{\text{dn}^4}(4p\varpi/k)$	$y^4 + \frac{6(x-x')}{xx'} y^3 + \frac{12}{x^2x'^2} y^2 + \frac{8(x-x')}{x^3x'^3} - \frac{3}{x^3x'^3} = 0$

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On Linearoid Differential Equations.

By Dr. E. J. Wilczynski.

The fundamental notions of the theory of linear differential equations can be applied to a very large category of non-linear differential equations. In a former paper* I have shown, in general, how this may be done, the existence of the differential equations, however, not being demonstrated. Moreover, the point of view in that paper was somewhat different, and the results obtained there are put into a clearer light if taken in connection with the present, which, however, is itself nothing but a reconnoissance upon, what appears to me, a new field of great promise.

I have ventured to call the differential equations, whose existence will be proven in this paper, by a new name. It would be very inconvenient to characterize them in every case by the enumeration of their properties. The name Linearoid suggests at once their relation to linear differential equations.

§1.—*Existence of Linearoid Differential Equations.*

By a system of linearoid differential equations we understand a system of differential equations

$$D_i(y_1, y_2, \ldots, y_n) = 0 \qquad (i = 1, 2, \ldots, n) \quad (1)$$

with the following property: Let $y_1, y_2, \ldots, y_n$ represent a system of particular solutions; then the general solutions of (1) are obtained in the form

$$\eta_i = \sum_{k=1}^{n} \phi_{ik}(x; a_1, a_2, \ldots, a_r) y_k, \qquad (i = 1, 2, \ldots, n) \quad (2)$$

where $a_1, a_2, \ldots, a_r$ are arbitrary constants and ϕ_{ik} are uniform functions with respect to x.

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* E. J. Wilczynski, "On Systems of Multiform Functions which belong to a Group of Linear Substitutions with Uniform Coefficients." Amer. Jour. Math., April, 1899.

Our equations may then be said to have fundamental solutions. However, there is this difference between our equations and those commonly said to have fundamental solutions, that in the latter the general integrals are

$$\eta_i = \Phi_i(y_1, \ldots, y_n; a_1, \ldots, a_r),$$

where Φ_i is independent of x. The linearoid equations are very general, and in fact if we did not suppose that ϕ_{ik} are uniform functions of x, would comprehend a very large class of differential equations.

It is of course assumed that $\eta_1, \ldots, \eta_n$, the general solutions, cannot be expressed in the form (2) in terms of less than n functions $y_1, \ldots, y_n$ or less than r constants, so that r is the order of the system, and there will be no relation verified of the form

$$\sum_{k=1}^{n} \phi_{ik}(x; b_1, b_2, \ldots, b_r)\, y_k = 0, \tag{3}$$

with non-vanishing coefficients. For if there were, $\eta_1, \ldots, \eta_n$ could be expressed in terms of only $n - 1$ y's.

Assuming no relation of the form (3) to be verified, $y_1, \ldots, y_n$ will be called a *fundamental system.*

Equations (2) *then define an r-parameter group G of continuous transformations.*

For give special but arbitrary values to $a_1, \ldots, a_r$. Then $\eta_1, \ldots, \eta_n$ are solutions of (1), provided $y_1, \ldots, y_n$ are such solutions. Then

$$\zeta_i = \sum_{k=1}^{n} \phi_{ik}(x; b_1, \ldots, b_r)\, \eta_k \qquad (i = 1, \ldots, n)$$

are also solutions of (1), whatever the values of the constants b_i may be. Therefore, since (2) gives the general solutions of (1), it must be possible to find $c_1, \ldots, c_r$ in terms of $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ such that

$$\zeta_i = \sum_{k=1}^{n} \phi_{ik}(x; c_1, \ldots, c_r)\, y_k. \qquad (i = 1, 2, \ldots, n)$$

But this proves the theorem.

If $y_1, \ldots, y_n$ form a fundamental system, $\eta_1, \ldots, \eta_n$ will also form a fundamental system, if the determinant

$$\Delta = |\phi_{ik}(x; a_1, a_2, \ldots, a_r)|$$

does not vanish identically.

For suppose an equation of the form

$$\sum_{k=1}^{n} \phi_{ik}(x;\ \lambda_1, \ldots\ldots, \lambda_r)\, y_k = 0$$

were verified. Then, owing to (2),

$$\sum_{k=1}^{n} \sum_{h=1}^{n} \phi_{ik}(x;\ \lambda_1, \lambda_2, \ldots\ldots, \lambda_r)\, \phi_{kh}(x;\ a_1, a_2, \ldots\ldots, a_r)\, y_h = 0;$$

but the coefficient of y_h can be written in the form $\phi_{ih}(x;\ c_1, \ldots\ldots, c_r)$ owing to the group property. Therefore, it must vanish, i. e.

$$\sum_{k=1}^{n} \phi_{ik}(x;\ \lambda_1, \ldots\ldots, \lambda_r)\, \phi_{kh}(x;\ a_1, \ldots\ldots, a_r) = 0, \qquad (h = 1, 2, \ldots\ldots, n)$$

whence, since $\Delta \neq 0$, it follows that

$$\phi_{ik}(x;\ \lambda_1, \ldots\ldots, \lambda_r) = 0. \qquad (k = 1, 2, \ldots\ldots, n)$$

Therefore, $\eta_1, \ldots\ldots, \eta_n$ form a fundamental system.

We have assumed the functions $\phi_{ik}(x;\ a_1, \ldots\ldots, a_r)$ to be uniform with respect to x. Let them be analytic functions of $a_1, \ldots\ldots, a_r$ likewise. Then we may assume that $\phi_{ik}(x;\ a_1, \ldots\ldots, a_r)$ can be developed in a series of powers of $a_1, \ldots\ldots, a_r$ convergent in a certain domain. The coefficient of a_λ in this development, or

$$\left(\frac{\partial \phi_{ik}}{\partial a_\lambda}\right)_0 = \psi_{ik}^{(\lambda)}(x),$$

is a uniform function of x. The infinitesimal transformations of the group G are therefore

$$U_\lambda f = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i} \sum_{k=1}^{n} \psi_{ik}^{(\lambda)}(x)\, y_k, \qquad (\lambda = 1, 2, \ldots\ldots, r) \quad (4)$$

or putting

$$\xi_{\lambda i} = \sum_{k=1}^{n} \psi_{ik}^{(\lambda)}(x)\, y_k, \quad U_\lambda f = \sum_{i=1}^{n} \xi_{\lambda i} \frac{\partial f}{\partial y_i}. \qquad (5)$$

Suppose for a moment that the r equations $U_\lambda f = 0$ are all independent. Denote by $G^{(s)}$ the group obtained by extending G s times. Then the r equations $U_\lambda^{(s)} f = 0$ are also independent. If $(s+1)n - r = n$, then $G^{(s)}$ will have just n independent invariants, and $sn = r$; i. e. the total order of these differential invariants of G is just equal to the number r of parameters in the group.

For every one of these differential invariants contains derivatives of order s. Otherwise there would be invariants of $G^{(s-1)}$, which is impossible, since $sn - r = 0$.

Suppose now that $sn - r = l$, where $l > 0$ but $< n$. Then $G^{(s-1)}$ has just $l < n$ invariants and $G^{(s)}$ has $n + l$. Each of the l invariants $\vartheta_1, \ldots, \vartheta_l$ of $G^{(s-1)}$ contains derivatives of order $s - 1$, and is also an invariant of $G^{(s)}$. Their derivatives $\vartheta_1', \ldots, \vartheta_l'$ are also invariants of $G^{(s)}$, and, moreover, they are independent of each other and $\vartheta_1, \ldots, \vartheta_l$. For suppose we had for instance

$$\vartheta_1' = \frac{d\vartheta_1}{dx} = \phi(\vartheta_1, \vartheta_2, \ldots, \vartheta_l;\ \vartheta_2', \ldots, \vartheta_l';\ x).$$

Since ϑ_1' is a total derivative, ϕ must also be a total derivative, so that

$$\frac{d\vartheta_1}{dx} = \frac{d}{dx}\psi(y_1, y_2, \ldots y_n;\ \ldots;\ y_1^{(s-1)}, \ldots, y_n^{(s-1)};\ x),$$

no derivatives of order s occurring in ψ, as that would give derivatives of order $s + 1$ in $\frac{d\vartheta_1}{dx}$. By integration

$$\vartheta_1 = \psi(y_1;\ \ldots;\ y_n;\ \ldots;\ y_1^{(s-1)} \ldots y_n^{(s-1)};\ x) + \text{const.}$$

But ϑ_1 is invariant under $G^{(s-1)}$. The same must therefore be true of ψ, whence

$$\vartheta_1 = \psi = \chi(\vartheta_1, \vartheta_2, \ldots, \vartheta_l;\ x).$$

But $\vartheta_1, \ldots, \vartheta_l$ being independent, χ and therefore ψ must reduce to ϑ_1, so that the supposed relation

$$\frac{d\vartheta_1}{dx} = \frac{d\psi}{dx}$$

reduces to the identity $\vartheta_1' = \vartheta_1'$.

Now let us omit the invariants $\vartheta_1', \ldots, \vartheta_l'$ and take the other n. They are independent, and none of them can be obtained from another in the set by differentiation. The total order of the system will be $(n - l)s + l(s - 1) = ns - l$. But we had $sn - r = l$, so that $sn - l = r$. In this case also the total order of the system of invariants is equal to the number of parameters in the group. Moreover, any other system of invariants of $G^{(s)}$ is of higher order than r, if it contains no invariants which can be obtained from one of the others by differentiation.

Suppose now that only $q < r$ of the equations $U_\lambda f = 0$ are independent. Then if $(s + 1)n - r = n + l$, $G^{(s)}$ will have at least $n + l$, and $G^{(s-1)}$ at least l

invariants. Thus we can as before construct at least one system of n independent invariants of total order r. For if $G^{(s-1)}$ has $\lambda \geqq l$ independent invariants, just λ of the invariants of $G^{(s)}$ are the derivatives of these. Suppress l of the latter and replace them by the corresponding l invariants of $G^{(s-1)}$. Take any $n-l$ of the other invariants of $G^{(s)}$. The order of such a system is $l(s-1)+(n-l)s = ns - l$. But $ns - r = l$, so that

$$ns - l = r.$$

In every case then, at least one system of independent differential invariants of total order r can be found. Let $\vartheta_1, \ldots, \vartheta_n$ denote such a system, in which $\vartheta_1, \ldots, \vartheta_n$ are independent, and also of such a nature that none of these expressions can be formed from one of the others by differentiation. Putting then

$$\vartheta_1 = r_1(x), \ldots, \vartheta_n = r_n(x), \tag{6}$$

where $r_k(x)$ are functions of x alone, we have a system of differential equations which remains unaltered by every transformation of the r-parameter group G. Therefore, if $y_1, \ldots, y_n$ constitute a fundamental system, the expressions

$$\eta_i = \sum_{k=1}^{n} \phi_{ik}(x;\, a_1, \ldots, a_r)\, y_k \quad (i = 1, 2, \ldots, n) \tag{7}$$

are also solutions of (6). Moreover, they are the general solutions, for they contain r arbitrary constants, and r is the order of the system.

This completes the proof of the existence theorem. *If the equations* (7) *form an r-parameter group, there always exists at least one system of differential equations of the r^{th} order, whose general solution is given by these equations, $y_1, \ldots, y_n$ being supposed to form a fundamental system.*

It must be noticed, however, that these differential equations are not necessarily algebraic, or still less, rational. *The linearoid differential equations have fixed branch-points*, i. e. the position of the branch-points is independent of the values of the constants of integration.

For, according to (7), since ϕ_{ik} are uniform functions of x, the general solutions $\eta_1, \ldots, \eta_n$ have the same branch-points as the special solution $y_1, \ldots, y_n$; i. e. the position of the branch-points is the same for all solutions, and therefore independent of the constants of integration.

We have

$$\xi_{\lambda k} = \psi_{k1}^{(\lambda)}(x)\, y_1 + \psi_{k2}^{(\lambda)}(x)\, y_2 + \ldots\ldots + \psi_{kn}^{(\lambda)}(x)\, y_n,$$

$$U_\lambda f = \sum_{k=1}^{n} \frac{\partial f}{\partial y_k}\, \xi_k \qquad (\lambda = 1, 2, \ldots\ldots, r)$$

and

$$Vf = 0\, \frac{\partial f}{\partial x}$$

as the infinitesimal transformations of the group G, since the variable x is transformed only by the identical transformation. Such transformations generate a group if the commutators $(U_\lambda,\ U_\mu)$ can be expressed linearly in terms of $U_1, \ldots\ldots, U_r$ with constant coefficients, i. e. with coefficients independent of $y_1, \ldots\ldots, y_n$, the parameters and x. Of course

$$(U_\lambda,\ V) = 0.$$

We have further

$$(U_\lambda,\ U_\mu) = \sum_{i=1}^{n} \left[U_\lambda(\xi_{\mu i}) - U_\mu(\xi_{\lambda i})\right] \frac{\partial f}{\partial y_i}.$$

But

$$U_\lambda(\xi_{\mu i}) = \sum_{k=1}^{n} \left(\psi_{k1}^{(\lambda)}(x)\, y_1 + \ldots\ldots + \psi_{kn}^{(\lambda)}(x)\, y_n\right) \psi_{ik}^{(\mu)}(x),$$

$$U_\mu(\xi_{\lambda i}) = \sum_{k=1}^{n} \left(\psi_{k1}^{(\mu)}(x)\, y_1 + \ldots\ldots + \psi_{kn}^{(\mu)}(x)\, y_n\right) \psi_{ik}^{(\lambda)}(x),$$

Thus we obtain

$$\begin{aligned}(U_\lambda,\ U_\mu) &= \sum_{i=1}^{n} \frac{\partial f}{\partial y_i} \sum_{k=1}^{n} \left[\left(\psi_{k1}^{(\lambda)}\psi_{ik}^{(\mu)} - \psi_{k1}^{(\mu)}\psi_{ik}^{(\lambda)}\right) y_1 + \ldots\ldots + \left(\psi_{kn}^{(\lambda)}\psi_{ik}^{(\mu)} - \psi_{kn}^{(\mu)}\psi_{ik}^{(\lambda)}\right) y_n\right] \\ &= \sum_{\nu=1}^{r} c_{\lambda\mu\nu} \sum_{i=1}^{n} \frac{\partial f}{\partial y_i} \left(\psi_{i1}^{(\nu)} y_1 + \psi_{i2}^{(\nu)} y_2 + \ldots\ldots + \psi_{in}^{(\nu)} y_n\right),\end{aligned}$$

if $c_{\lambda\mu\nu}$ denotes constant quantities.

For $f = y_i$, we obtain

$$\begin{aligned}&\sum_{k=1}^{n} \left[\left(\psi_{k1}^{(\lambda)}\psi_{ik}^{(\mu)} - \psi_{k1}^{(\mu)}\psi_{ik}^{(\lambda)}\right) y_1 + \ldots\ldots + \left(\psi_{kn}^{(\lambda)}\psi_{ik}^{(\mu)} - \psi_{kn}^{(\mu)}\psi_{ik}^{(\lambda)}\right) y_n\right] \\ &= \sum_{\nu=1}^{r} c_{\lambda\mu\nu} \left(\psi_{i1}^{(\nu)} y_1 + \psi_{i2}^{(\nu)} y_2 + \ldots\ldots + \psi_{in}^{(\nu)} y_n\right).\end{aligned}$$

$$(\lambda,\ \mu = 1, 2, \ldots\ldots, r;\ i = 1, 2, \ldots\ldots, n)$$

But as this must be an identity, we obtain by equating coefficients of y_σ

$$\sum_{k=1}^{n} [\psi_{k\sigma}^{(\lambda)}\psi_{ik}^{(\mu)} - \psi_{k\sigma}^{(\mu)}\psi_{ik}^{(\lambda)}] = \sum_{\nu=1}^{r} c_{\lambda\mu\nu}\psi_{i\sigma}^{(\nu)}. \tag{8}$$

$$(\lambda, \mu = 1, 2, \ldots, r;\ i, \sigma = 1. 2, \ldots, n)$$

Only *if n^2r functions $\psi_{i\sigma}^{(\nu)}(x)$ and r^3 constants $c_{\lambda\mu\nu}$ can be found verifying these relations will the r infinitesimal transformations $U_\lambda f$ and Vf generate an r-parameter linearoid group.* Of course, the constants $c_{\lambda\mu\nu}$ must also verify the relations

$$\sum_{\sigma=1}^{r} (c_{gk\sigma}c_{\sigma j\tau} + c_{kj\sigma}c_{\sigma g\tau} + c_{jg\sigma}c_{\sigma k\tau}) = 0$$

and

$$c_{gk\tau} + c_{kg\tau} = 0.$$

Suppose we have found n^2r such functions and r^3 such constants. In order that the group which $U_\lambda f$ and Vf generate may have coefficients which are uniform functions of x, the functions $\psi_{ik}^{(\lambda)}(x)$ must be uniform. But this condition is not sufficient. The finite equations of the group are found by integrating the simultaneous system

$$\frac{d\eta_k}{dt} = \sum_{\lambda=1}^{r} c_\lambda \xi_{\lambda k} = \sum_{\lambda=1}^{r} c_\lambda [\psi_{k1}^{(\lambda)}(x)\,\eta_1 + \psi_{k2}^{(\lambda)}(x)\,\eta_2 + \ldots + \psi_{kn}^{(\lambda)}(x)\,\eta_n],$$

or

$$\frac{d\eta_k}{dt} = \sum_{i=1}^{n} [c_1\psi_{ki}^{(1)}(x) + c_2\psi_{ki}^{(2)}(x) + \ldots + c_r\psi_{ki}^{(r)}(x)]\,\eta_i = \sum_{i=1}^{n} A_{ki}\,\eta_i, \tag{9}$$

$$(k = 1, 2, \ldots, n)$$

with the condition $\eta_k = y_k$ for $t = 0$. We have put

$$A_{ki} = c_1\psi_{ki}^{(1)}(x) + c_2\psi_{ki}^{(2)}(x) + \ldots + c_r\psi_{ki}^{(r)}(x). \tag{10}$$

$$(k, i = 1, 2, \ldots, n)$$

The quantities A_{ki} are independent of t. From the well-known theory of such systems, it follows that in order that the general integrals may be uniform functions of x, *the characteristic determinant*

$$|A_{ki} - \delta_{ki}\omega| = 0, \tag{11}$$

where $\delta_{ki} = 0,\ k \neq i;\ \delta_{kk} = 1$

must be reducible to a product of linear factors, in the field of the quantities A_{ki}, for all values of $c_1, c_2, \ldots, c_r$. This condition is necessary and sufficient. Moreover, *the coefficients of the finite equations of the groups are then uniform functions of the parameters* $c_1, \ldots, c_r$ *also.* Moreover, these formulæ show that *not only are the branch-points of linearoid differential equations fixed, but also the essentially singular points in Weierstrass's terminology.*

If the above condition is not fulfilled, these coefficients will be at the most n-valued functions of $x, c_1, \ldots, c_r$, provided that $\psi_{ik}^{(\lambda)}(x)$ are uniform functions.

§2.—*One-Parameter Groups, and certain r-Parameter Groups, all of whose Infinitesimal Transformations are Commutative.*

Let any infinitesimal transformation of the form

$$\left.\begin{aligned} Uf &= \sum_{k=1}^{n} \frac{\partial f}{\partial y_k}\,\xi_k, \\ \xi_k &= \psi_{k1}(x)\,y_1 + \psi_{k2}(x)\,y_2 + \ldots + \psi_{kn}(x)\,y_n \end{aligned}\right\} \tag{1}$$

be given, where $\psi_{ki}(x)$ are uniform functions of x. It will generate a one-parameter group linear in $y_1, \ldots, y_n$, with coefficients which are uniform functions of x, if the determinant

$$\begin{vmatrix} \psi_{11}(x) - \omega, & \psi_{12}(x) & , \ldots, & \psi_{1n}(x) \\ \psi_{21}(x) & , \psi_{22}(x) - \omega, & \ldots, & \psi_{2n}(x) \\ \cdots & \cdots & \cdots & \cdots \\ \psi_{n1}(x) & , \psi_{n2}(x) & , \ldots, & \psi_{nn}(x) - \omega \end{vmatrix} \tag{2}$$

reduces to a product of linear factors in the field of $\psi_{ik}(x)$.

If $\mathfrak{y}_1, \mathfrak{y}_2, \ldots, \mathfrak{y}_{n-1}$ be the $n-1$ independent solutions of $Uf = 0$, and $\mathfrak{y}_n$ a solution of $Uf = 1$, the variables $\mathfrak{y}_1, \ldots, \mathfrak{y}_n$ are transformed as follows:

$$\bar{x} = x, \quad \bar{\mathfrak{y}}_i = \mathfrak{y}_i \ (i = 1, 2, \ldots, n-1), \quad \bar{\mathfrak{y}}_n = \mathfrak{y}_n + t. \tag{3}$$

The equations

$$\mathfrak{y}_i(x, y_1, y_2, \ldots, y_n) = f_i(x), \quad \frac{d\mathfrak{y}_n}{dx} = f_n(x), \tag{4}$$

then form a system of differential equations such that if $y_1, \ldots, y_n$ form a fundamental system, the general solutions are given by

$$\eta_i = \sum_{k=1}^{n} \phi_{ik}(x, t)\, y_k, \qquad (i = 1, 2, \ldots, n) \qquad (5)$$

these being the finite equations of the group generated by (1).

Essentially the same method applies to an r-parameter group generated by r infinitesimal transformations of the form

$$U_\lambda f = f_\lambda(x)\, Uf. \qquad (\lambda = 1, 2, \ldots, n) \qquad (6)$$

These are all commutative, and have the same path-curves. The canonical form of the group becomes

$$\left.\begin{array}{ll} \bar{\mathfrak{y}}_i = \mathfrak{y}_i, & (i = 1, 2, \ldots, n-1) \\ \bar{\mathfrak{y}}_n = \mathfrak{y}_n + c_1 f_1(x) + c_2 f_2(x) + \ldots + c_r f_r(x), & \end{array}\right\} \qquad (7$$

$c_1, \ldots, c_r$ being arbitrary constants. Therefore $\mathfrak{y}_n$ is the general integral of a non-homogeneous linear differential equation of the r^{th} order whose fundamental integrals are $f_1(x), \ldots, f_r(x)$, By elimination, equations of the r^{th} order can be found for $y_1, y_2, \ldots, y_n$. But this elimination is not in general algebraically possible.

The functions $y_1, \ldots, y_n$, thus defined by differential equations, will, if the coefficients of these equations are uniform, have the following function-theoretic property. They are uniform except in the vicinity of certain points $a_1, \ldots, a_m$ in the plane, and when x describes a close curve around a_i, $y_1, \ldots, y_n$ will undergo a linear substitution with coefficients which are uniform functions of x. The group of linearoid substitutions thus obtained is a discreet numerable subgroup of the continuous group generated by $U_1 f, \ldots, U_r f$.

§3.—*The Group of Rotations.*

To give an example, let us consider the r-parameter group of rotations

$$\left.\begin{array}{l} \eta_1 = \cos \omega y_1 - \sin \omega y_2, \\ \eta_2 = \sin \omega y_1 + \cos \omega y_2, \\ \omega = c_1 f_1(x) + c_2 f_2(x) + \ldots + c_r f_r(x). \end{array}\right\} \qquad (1)$$

The canonical form of the group will be found by putting

$$r = \sqrt{y_1^2 + y_2^2}, \quad f = \text{arc tan}\, \frac{y_2}{y_1}.$$

The transformed group becomes

$$\rho = r, \quad \phi = f + c_1 f_1(x) + c_2 f_2(x) + \dots + c_r f_r(x),$$

so that ϕ is the general integral of a non-homogeneous linear differential equation of the r^{th} order,

$$\phi^{(r)} + p_1 \phi^{(r-1)} + \dots + p_r \phi = p,$$

of which f is a special solution, and such that $f_1(x), \dots, f_r(x)$ form a fundamental system of the homogeneous equation

$$\phi^{(r)} + p_1 \phi^{(r-1)} + \dots + p_r \phi = 0.$$

The uniform functions $f_1(x), \dots, f_r(x)$ being given, the coefficients p_λ of this equation, are easily determined. They are uniform functions of x. Further, $r^2 = y_1^2 + y_2^2 = s(x)$ may be chosen as a uniform function of x. For the determination of η_1 and η_2, we have then the two equations

$$\left.\begin{array}{l} \sum_{k=0}^{r} p_k \frac{d^{r-k}}{dx^{r-k}} \text{arc tan}\, \frac{\eta_2}{\eta_1} = p, \qquad p_0 = 1 \\ \eta_1^2 + \eta_2^2 = s(x), \end{array}\right\} \tag{3}$$

where we will, moreover, now assume that the coefficients p, p_k and $s(x)$ are rational functions of x. If y_1, y_2 form a fundamental system of (3), the general solution is given by (1).

If y_1, y_2 is a fundamental system of (3), let a be a branch-point of these functions. If x describes a circuit around a, y_1 and y_2 must undergo a linear substitution of the form (1). For all possible closed paths y_1 and y_2 will undergo a group of linear substitutions, which is a subgroup of (1).

In particular, let $f_\lambda(x) = x^\lambda$ for $\lambda = 0, 1, 2, \dots, r-1$. The differential equations then assume the simple form

$$\left.\begin{array}{l} \frac{d^r}{dx^r} \text{arc tan}\, \frac{\eta_2}{\eta_1} = r(x), \\ \eta_1^2 + \eta_2^2 = s(x) \end{array}\right\} \tag{4}$$

$r(x)$ and $s(x)$ being rational functions of x. Let a_i be a pole of $r(x)$ such that

$$r(x) = \sum_{i=1}^{m} \left[\frac{A_{ir}}{(x-a_i)^r} + \frac{A_{i,\,r-1}}{(x-a_i)^{r-1}} + \dots + \frac{A_{i1}}{x-a_i} \right] + r'(x),$$

where $r'(x)$ contains all terms of $r(x)$ which, after r-fold integration, give a rational integral. $r'(x)$ may then still contain any number of poles, but each of these is of order higher than r. Integrating r times, we obtain

$$\begin{aligned} \operatorname{arc\,tan} \frac{\eta_2}{\eta_1} &= \frac{1}{2\pi i} \sum_{i=1}^{m} \log(x-a_i)[\lambda_{i0} + \lambda_{i1}x + \dots + \lambda_{ir-1}x^{r-1}] + \rho(x) \\ &= \sigma(x), \end{aligned} \tag{5}$$

where $\rho(x)$ is a rational function of x. By properly choosing the constants A_{ik}, the constants λ_{ik} may be made to assume any arbitrarily assigned values, as may be easily verified. But if x makes a circuit around a_i, $\sigma(x)$ is increased by $\lambda_{i0} + \lambda_{i1}x + \dots + \lambda_{i,\,r-1}x^{r-1}$. Therefore,

$$\eta = \frac{\eta_2}{\eta_1} = \tan\sigma(x), \tag{6}$$

is uniform everywhere except in the vicinities of the points a_i. When λ makes a positive circuit around a_i, η suffers a projective substitution of the form

$$\bar{\eta} = \frac{\eta + \tan\phi_i(x)}{1 - \eta\tan\phi_i(x)},$$

where $\phi_i(x)$ are arbitrary polynomials of degree $r-1$. η_1 and η_2 undergo the corresponding homogeneous substitutions. In fact from (6) and (4) we find

$$\eta_1 = \sqrt{s(x)}\cos\sigma(x), \quad \eta_2 = \sqrt{s(x)}\sin\sigma(x). \tag{7}$$

If η_1 and η_2 are to have only the branch-points $a_1, \dots, a_m$ and no others, the rational function $s(x)$ must have no zeros or poles of odd order, and the fundamental substitutions otherwise arbitrary must verify the relation

$$A_1A_2 \dots A_m = 1,$$

or

$$\sum_{i=1}^{m} \lambda_{ik} = 0, \quad k \neq 0, \quad \sum_{i=1}^{m} \lambda_{i0} = 2\mu\pi,$$

where μ is an integer. If these relations were not verified, the point $x=\infty$ would introduce itself as a branch-point with the substitution A_{m+1} such that

$$A_1 A_2 \ldots . A_m A_{m+1} = 1 .$$

It is easy to verify that η_1 and η_2 are both solutions of the differential equation

$$\left[\frac{d^{r-1}}{dx^{r-1}} \frac{\frac{1}{2} s'(x) y - s(x) \frac{dy}{dx}}{2s(x)\sqrt{s(x)-y^2}}\right]^2 = r(x)^2 .$$

Suppose for simplicity that there are only three branch-points 0, 1 and ∞. Then, according to a general theorem of M. Poincaré, if we put

$$x = \phi(\zeta),$$

where $\phi(\zeta)$ denotes the elliptic modular function,

$$\eta = \tan \sigma [\phi(\zeta)] = \psi(\zeta)$$

will be a uniform function of ζ. Moreover, if ζ undergoes a substitution of the group of the modular function, η will suffer a projective substitution whose coefficients are uniform functions of x and therefore automorphic uniform functions of ζ. Of course we obtain in the same way also a system of two uniform functions of ζ which, when ζ undergoes a substitution of the modular group, undergoes an orthogonal homogeneous substitution whose coefficients are automorphic functions of ζ.

This is, so far as I know, the first example of a new kind of homomorphic function, excepting the simple case of Poincaré's Thetafuchsian and Thetakleinian functions.

Generally there are thus brought to our attention systems of uniform functions of a variable ζ which undergo a group of homogeneous linear or projective substitutions with coefficients which are automorphic functions of ζ, when ζ suffers a group of projective substitutions with constant coefficients.

The existence of an extensive class of such functions can be established by the methods of M. Poincaré making use of the Zetafuchsian series, and numerous other examples can be found.

§4.—*Conclusion.*

In my previous paper in the American Journal, a theorem is proven which may appear to be in contradiction with some of the results of the present paper. It is there shown that functions behaving as the solutions of linearoid differential equations, verify homogeneous differential equations, under certain conditions. The equations which we have found, however, are not homogeneous. The reason for this is simply that in the former paper the domain of rationality was formed by the coefficients of the fundamental substitutions, and here a different realm is employed.

Our results serve only as an introduction to what seems to me quite a new field. An extensive class of differential equations comes before us, for which a general theory becomes possible. The explicit determination of linearoid equations and their closer discussion would appear to be a matter of great importance and interest. This can in many cases be done without any integration. It is true, as will be shown in a subsequent paper, that most of these equations can be reduced to a combination of linear differential equations and quadratures. This, of course, is in itself interesting, and shows that only the simplest cases of Λ functions are obtained by the methods of this paper, which is but an application of *Lie's* theory of finite continuous groups, which can be at once generalized for other groups. It is essential, however, for the function-theoretic application that the independent variable x is not transformed at all.

UNIVERSITY OF CALIFORNIA, BERKELEY, CAL., *February* 21, 1899.

On the Roots of a Determinantal Equation.

By W. H. Metzler.

1. In the American Journal of Mathematics* Dr. Thomas Muir made use of Sylvester's proof of the reality of the roots of Lagrange's determinantal equation to prove the theorem

The n^{thic} *equation,*

$$\begin{vmatrix} 11-x & 12 & 13 & \dots \\ 21 & 22-x & 23 & \dots \\ 31 & 32 & 33-x & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0$$

will have all its roots real if in the case of every pair μ, ν *of the* $n-1$ *indices* $2, 3, 4, \dots, n$, *we have*

$$1\mu \, . \, \mu\nu \, . \, \nu 1 = \mu 1 \, . \, \nu\mu \, . \, 1\nu$$

and

$$1\mu \, . \, \mu 1 = + \, .$$

There is another theorem quite similar to this which may be proven in a precisely similar manner.

It is as follows:

The n^{thic} *equation*

$$\begin{vmatrix} 11-x & 12 & 13 & \dots \\ 21 & 22-x & 23 & \dots \\ 31 & 32 & 33-x & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0$$

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* Vol. XIX, No. 4, pp. 812-818.

*will have all its roots pure imaginary** *if in the case of every pair* μ, ν *of the* $n-1$ *indices* $2, 3, 4, \ldots, n$, *we have*

$$1\mu . \mu\nu . \nu 1 = -\mu 1 . \nu\mu . 1\nu,$$

$$1\mu . \mu 1 = -,$$

and

$$\lambda\lambda = 0. \qquad (\lambda = 1, 2, 3, \ldots, n)$$

2. In art. 5 of his paper Dr. Muir shows that, in any determinant in which the specified conditions mentioned in the first theorem are true, every term is equal to its conjugate. In exactly the same way it may be shown that, in any determinant in which the specified conditions mentioned in the second theorem are true, every term is equal to its conjugate if the order of the determinant is even and to the negative of its conjugate if the order of the determinant is odd. Consequently, in case the order of the determinant is odd, conjugate terms cancel each other. Again, when the order is odd, self-conjugate terms contain at least one constituent from the principal diagonal and therefore vanish.

It follows that every determinant of odd order, in which the specified conditions of the second theorem are true, vanishes.

3. If now, following Dr. Muir, we consider the equation of the fourth degree, the proof of the second theorem proceeds precisely the same as that given of the first up to the point where the terms of the quartic in x^2 are shown to be alternately positive and negative.

Examining these terms under the conditions of the second theorem, it will be seen, on making use of the results of art. 2, that the coefficient of $-x^6$ is negative, the coefficient of x^4 is positive, the coefficient of $-x^2$ is negative, and the independent term is positive. The quartic, therefore, has all its terms positive, and consequently can have no real positive root. It can have no imaginary roots, for the specified conditions of the first theorem are true in the determinantal equation $f(x)f(-x) = 0$. Hence the roots of the quartic in x^2 are all negative, and therefore the quartic in x has all its roots pure imaginary.

SYRACUSE UNIVERSITY, *March*, 1898.

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* If n is odd and $= 2\rho + 1$, one root will be zero and the other 2ρ will be pure imaginary.

Non-Quaternion Number-Systems Containing No Skew Units.

By Dr. G. P. Starkweather.

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§1.

In §2 is first given a brief statement of a few important properties of number-systems in general. Next is given a proof of a statement made by Scheffers as to the possibility, in the special class of number-systems here considered, of a selection of units having certain simple multiplicative properties (see p. 371).

In §3 it is shown that the units can be so chosen as to give in general a very much simplified form of multiplication table, and a method is given for deriving systems of the type considered in n units from those in $(n-1)$ units (p. 376).

In §4 is given a theorem on nilfactors (p. 377).

In §5 application of the principles deduced is made to systems the degree of whose characteristic equation is two less than the number of units. Certain general theorems are proved (p. 379), and the systems are reduced to a few typical forms having some peculiar properties (pp. 380, 381, 382).

In §6 the parameters of the systems discussed in §5 are specialized, so far as possible, for the case when the number of units exceeds six, and a table of all the possible non-equivalent forms is given (pp. 385, 386).

§2.

It has been shown by Scheffers* that complex number-systems in n units can be divided into two distinct classes. In any system of the first class, called, after its best-known representative, the quaternion class, there exist three quantities, e_1, e_2, e_3, between which and the modulus, or idemfactor, no linear relation exists, such that

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* "Complexe Zahlensysteme," Mathematische Annalen, XXXIX, pp. 306, 310.

$$\left.\begin{aligned} e_1 e_2 - e_2 e_1 &= 2e_3, \\ e_2 e_3 - e_3 e_2 &= 2e_1, \\ e_3 e_1 - e_1 e_3 &= 2e_2. \end{aligned}\right\} \tag{1}$$

For every number-system of the second class, to which the name non-quaternion is given, it is possible to choose as units quantities

$$u_1 \dots u_r \quad \eta_1 \dots \eta_s$$

which have the following multiplicative properties: $u_i u_j$ and $u_j u_i$, $j \not> i$, are linear functions of $u_1 \dots u_{j-1}$. $\eta_i^2 = \eta_i$. $\eta_i \eta_k = 0$, $i \neq k$. $\eta_i u_k$ is zero except for one value of i, say λ_k, when it equals u_k, and similarly $u_k \eta_i$ is zero except for one value of i, say μ_k, when it equals u_k. If $\mu_k \neq \lambda_k$, the unit u_k is said to be skew, otherwise even. This form is called the regular form, and no quaternion system can be put in it, nor does any non-quaternion system contain quantities satisfying the equations (1).

If we consider now non-quaternion systems without skew units, if there be more than one of the quantities η, the system can be reduced to a sum of systems containing each only one η.* Therefore, it is assumed that in the systems here considered there are $(n-1)$ of the units u and only one η, which is the modulus. Any number

$$x = a_1 u_1 + \dots a_{n-1} u_{n-1} + \xi\eta$$

(where $a_1 \dots a_{n-1}$, ξ are ordinary complex quantities) satisfies the equation†

$$(x - \xi\eta)^v = 0,$$

where $v \not> n$. This is the characteristic equation. If $v = n - \delta$, δ may be called the deficiency of the system.

It follows from this equation that every number σ formed from the units $u_1 \dots u_{n-1}$ satisfies the equation $\sigma^{n-\delta} = 0$. It must be possible to choose σ such that $\sigma^{n-\delta-1} \neq 0$, else the characteristic equation would be of lower degree.

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*Ibid., p. 328.

†Ibid., p. 316.

The $(n-\delta-1)$ quantities $\sigma, \sigma^2 \dots . \sigma^{n-\delta-1}$ are all linearly independent, for suppose

$$a_1\sigma^{n-\delta-1} + a_2\sigma^{n-\delta-2} + \dots . = 0 .$$

By multiplying enough times by σ we can make all the terms vanish but one, and hence have $\sigma^k = 0$, where $k < n - \delta$, contrary to hypothesis. *Hence, we can use* $\sigma^{n-\delta-1}$, $\sigma^{n-\delta-2} \dots . \sigma$ *as* $(n-\delta-1)$ *new units,* $w_1, w_2 \dots . w_{n-\delta-1}$, *where* $w_a = \sigma^{n-\delta-a}$. The multiplication of the w's is very simple, following from that of the σ's, and will be regular in Scheffers' sense. In fact, $w_i w_k = w_{i+k-n+\delta}$ *or* 0, *according as* $i + k - n + \delta > 0$ *or* $\not> 0$.

These units, with η, make a total of $(n-\delta)$. The remaining δ may be selected from the u's in the following way:

$$\sigma = a_{1,1}u_1 + \dots . a_{1,n-1}u_{n-1}.$$

From the multiplicative properties of the u's, σ^2 can contain no u of as high an index as occurs in σ, similarly for σ^3 with respect to σ^2, etc. Solve each equation

$$\sigma^k = a_{k,1}\, u_1 + \dots .$$

for the u of highest index occurring therein. Each of these u's is, therefore, expressible in a σ^k and u's of lower index, that is, in a w and u's of lower index. Since there are $(n-1)$ u's and only $(n-\delta-1)$ equations, there are δ u's which are not so expressed. These, which will be denoted by $u_{z_1}, u_{z_2} \dots . u_{z_\delta}$, $z_1 < z_2 \dots . < z_\delta$, will be taken for the remaining δ units. This is possible, for any u, u_j, other than these, can be expressed, we have seen, in terms of a w, w_k, and u's of lower index than u_j, hence ultimately in terms of $w_1 \dots . w_k$ and $u_{z_1} \dots . u_{z_a}$, where u_{z_a} is that one of u_{z_1}; $u_{z_2} \dots . u_{z_\delta}$ immediately preceding u_j.

The multiplication of the w's we have remarked to be regular. *If we write the units* $w_1 \dots . w_{n-\delta-1}, u_{z_1} \dots . u_{z_\delta}$ *in such order that no* w *preceding* u_{z_a} *contains a* u *of as high index as* z_a, *and every* w *following* u_{z_a} *contains a* u *of higher index than* z_a, *which is evidently possible, the multiplication of the new units will be entirely regular.* For if we consider $u_{z_a} w_k$ and $w_k u_{z_a}$, where w_k lies beyond u_{z_a}, the product expressed in u's can contain no u of higher index than $(z_a - 1)$, and the product is accordingly expressible in w's and u_z's preceding u_{z_a}. Similarly for $u_{z_a} w_k$ and $w_k u_{z_a}$, where w_k occurs before u_{z_a}, and also for $u_{z_a} u_{z_\beta}$.

This verifies a statement made without proof by Scheffers,* in considering the cases $\delta = 1$ and $\delta = 2$, that the δ units in addition to $w_1 \dots w_{n-\delta-1}$, η could be so chosen as to make the table regular, although their position relative to the w's is unknown.

§3.

It is now proposed to show that by abandoning in part the regular form, the multiplication table can be simplified in certain cases. The methods are an extension of those used by Scheffers in considering the case $\delta = 1$.

The units $w_1 \dots w_{n-\delta-1}$, $u_{s_1} \dots u_{s_\delta}$ being regular in some order, in which the order of $u_{s_1} \dots u_{s_\delta}$ is, however, unchanged, u_{s_δ} occurs in none of their products. Therefore, the system $w_1 \dots w_{n-\delta-1}, u_{s_1} \dots u_{s_{\delta-1}}$, η is unchanged by the deletion of u_{s_δ}, and is a system of $(n-1)$ units. Since $w_{n-\delta-1} = \sigma$ is in this system, the characteristic equation is the same as before; hence this system of $(n-1)$ units is of deficiency $(\delta - 1)$. It will now be assumed that any system of deficiency $(\delta - 1)$ can, by replacing u_{s_β} by

$$\tau_\beta = u_{s_\beta} + a_{2,\beta} w_2 + \dots a_{(n-\delta-1),\beta} w_{n-\delta-1}$$
$$\beta = 1, 2 \dots (\delta - 1)$$

be put in a form having the following properties: $\tau_\beta w_{n-\delta-k}$ and $w_{n-\delta-k} \tau_\beta$ are zero if $k > \beta$, contain only w_1 if $k = \beta$, while if $k < \beta$ they are linear functions of $w_1 \dots w_{\beta-k+1}$, $\tau_1 \dots \tau_{\beta-k}$. We wish to prove that the same can be done for systems of deficiency δ. By the notation $(x, y, z \dots)$ will be meant a linear function of $x, y, z \dots$.

Now,

$$u_{s_\delta} w_{n-\delta-1} = u_{s_\delta}(u_1, \dots u_{n-1}) = (u_1, u_2 \dots u_{s_\delta - 1}).$$

All of these last u's are expressible in w's and $u_{s_1}, u_{s_2} \dots u_{s_{\delta-1}}$ (see p, 371), hence, in w's and $\tau_1, \tau_2 \dots \tau_{\delta-1}$. So

$$u_{s_\delta} w_{n-\delta-1} = c_1 w_1 + \dots c_{n-\delta-1} w_{n-\delta-1} + d_1 \tau_1 + \dots d_{\delta-1} \tau_{\delta-1} \tag{2}$$

or

$$u_{s_\delta} \sigma = c_1 \sigma^{n-\delta-1} + \dots c_{n-\delta-1} \sigma + d_1 \tau_1 + \dots d_{\delta-1} \tau_{\delta-1}. \tag{3}$$

Similarly,

$$w_{n-\delta-1} u_{s_\delta} = c'_1 w_1 + \dots c'_{n-\delta-1} w_{n-\delta-1} + d'_1 \tau_1 + \dots d'_{\delta-1} \tau_{\delta-1}. \tag{2'}$$

$$\sigma u_{s_\delta} = c'_1 \sigma^{n-\delta-1} + \dots c'_{n-\delta-1} \sigma + d'_1 \tau_1 + \dots d'_{\delta-1} \tau_{\delta-1}. \tag{3'}$$

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* Ibid, pp. 338, 340, 341.

Case I. $n \not> 2\delta + 1$.

Let $\tau_\delta = u_{z_\delta} + \lambda_2 w_2 + \ldots . \lambda_{n-\delta-1} w_{n-\delta-1}$ where $\lambda_2 \ldots \lambda_{n-\delta-1}$ are arbitrary. Then,

$$\tau_\delta w_{n-\delta-1} = u_{z_\delta} w_{n-\delta-1} + \lambda_2 w_1 + \ldots . \lambda_{n-\delta-1} w_{n-\delta-2} = (c_1 + \lambda_2) w_1$$
$$+ \ldots . (c_{n-\delta-2} + \lambda_{n-\delta-1}) w_{n-\delta-2} + c_{n-\delta-1} w_{n-\delta-1} + d_1 \tau_1 + \ldots . d_{\delta-1} \tau_{\delta-1}. \quad (4)$$

Multiply this by σ^{k-1}, obtaining

$$\tau_\delta w_{n-\delta-k} = e_1 w_1 + \ldots . c_{n-\delta-1} w_{n-\delta-k} + (d_1 \tau_1 \sigma^{k-1} + \ldots . d_{\delta-1} \tau_{\delta-1} \sigma^{k-1}).$$

Now by the law assumed for the multiplication of $\tau_1 \ldots . \tau_{\delta-1}$ with the w's, no higher w can occur from the parenthesis than $w_{\delta-k+1}$ and no higher τ than $\tau_{\delta-k}$. And since $n \not> 2\delta + 1$, $n - \delta - k \not> \delta - k + 1$, and accordingly $\tau_\delta w_{n-\delta-k}$ has as its highest w, $w_{\delta-k+1}$, and as its highest τ, $\tau_{\delta-k}$. If $k > \delta$ the terms all vanish. A similar proof applies to $w_{n-\delta-k}\tau_\delta$. Hence the multiplication of τ_δ with the w's is according to the law assumed for $\tau_1 \ldots . \tau_{\delta-1}$.

From (2′)

$$w_{n-\delta-1} \tau_\delta = (c'_1 + \lambda_2) w_1 + \ldots . (c'_{n-\delta-2} + \lambda_{n-\delta-1}) w_{n-\delta-2}$$
$$+ c'_{n-\delta-1} w_{n-\delta-1} + d'_1 \tau_1 + \ldots . d'_{\delta-1} \tau_{\delta-1}. \quad (4')$$

Hence, the λ's can be so chosen as to make the coefficient of w_i occurring in $\tau_\delta w_{n-\delta-1}$ the negative of that occurring in $w_{n-\delta-1} \tau_\delta$, except for $i = n - \delta - 1$.

Case II. $n > 2\delta + 1$.

Multiply (3) by σ^δ. Since $n - \delta > \delta + 1$, $\sigma^{\delta+1} \neq 0$. We have then

$$u_{z_\delta} \sigma^{\delta+1} = c_{\delta+1} \sigma^{n-\delta-1} + \ldots . c_{n-\delta-1} \sigma^{\delta+1} + (d_1 \tau_1 + \ldots . d_{\delta-1} \tau_{\delta-1}) \sigma^\delta. \quad (5)$$

But by the assumed law of multiplication the last term is zero. Multiplying by σ^{i-1},

$$u_{z_\delta} \sigma^{\delta+i} = c_{\delta+i} \sigma^{n-\delta-1} + \ldots \ldots c_{n-\delta-1} \sigma^{\delta+i}.$$

So from (5),

$$u^2_{z_\delta} \sigma^{\delta+1} = g_1 \sigma^{n-\delta-1} + \ldots . g_{n-\delta-2} \sigma^{\delta+1} + c^2_{n-\delta-1} \sigma^{\delta+1}. \quad (6)$$

But

$$u^2_{z_\delta} = (u_1 \ldots . u_{z_\delta - 1}) = (w_1 \ldots . w_{n-\delta-1}, u_{z_1} \ldots . u_{z_{\delta-1}})$$
$$= (\sigma, \sigma^2 \ldots . \sigma^{n-\delta-1}, \tau_1 \ldots . \tau_{\delta-1}).$$

Multiplying by $\sigma^{\delta+1}$, since $\tau_1 \ldots . \tau_{\delta-1}$ all make zero into $\sigma^{\delta+1}$, we find that $u_{z_\delta}^2 \sigma^{\delta+1}$ does not contain $\sigma^{\delta+1}$, hence, comparing with (6), $c_{n-\delta-1} = 0$.

Therefore, equation (3) becomes

$$u_{z_\delta}\sigma = c_1\sigma^{n-\delta-1} + \ldots . c_{n-\delta-2}\sigma^2 + d_1\tau_1 + \ldots . d_{\delta-1}\tau_{\delta-1}. \tag{7}$$

Similarly,

$$\sigma u_{z_\delta} = c_1'\sigma^{n-\delta-1} + \ldots . c_{n-\delta-2}'\sigma^2 + d_1'\tau_1 + \ldots . d_{\delta-1}'\tau_{\delta-1}. \tag{7'}$$

If $n = 2\delta + 2$ the method of Case I can now be followed. If $n > 2\delta + 2$, $n - \delta > \delta + 2$, and $\sigma^{\delta+2} \neq 0$. Multiplying (7) by σ^δ left-handed and (7′) by $\sigma^{\delta-1}$ left-handed and σ right-handed, and equating,

$$c_{\delta+1}\sigma^{n-\delta-1} + \ldots . c_{n-\delta-2}\sigma^{\delta+2} = c_{\delta+1}'\sigma^{n-\delta-1} + \ldots . c_{n-\delta-2}'\sigma^{\delta+2} + d_{\delta-1}'\sigma^{\delta-1}\tau_{\delta-1}\sigma.$$

But $\sigma^{\delta-1}\tau_{\delta-1}$ can contain only w_1 or $\sigma^{n-\delta-1}$. Hence the last term vanishes. Since the σ's are linearly independent, we must, therefore, have

$$c_{\delta+1}' = c_{\delta+1}, \ldots . c_{n-\delta-2}' = c_{n-\delta-2}.$$

Accordingly, if we substitute

$$\tau_\delta = u_{z_\delta} + \lambda_1\sigma^{n-\delta-2} + \ldots . \lambda_\delta\sigma^{n-2\delta-1} - c_{\delta+1}\sigma^{n-2\delta-2} - \ldots . - c_{n-\delta-2}\sigma,$$

where the λ's are arbitrary, there results

$$\tau_\delta w_{n-\delta-1} = (c_1 + \lambda_1)w_1 + \ldots . (c_\delta + \lambda_\delta)w_\delta + d_1\tau_1 + \ldots . d_{\delta-1}\tau_{\delta-1}, \tag{8}$$

and likewise

$$w_{n-\delta-1}\tau_\delta = (c_1' + \lambda_1)w_1 + \ldots . (c_\delta' + \lambda_\delta)w_\delta + d_1'\tau_1 + \ldots . d_{\delta-1}'\tau_{\delta-1}. \tag{8'}$$

Multiplying (8) by $\sigma^{k-1} = w_{n-\delta-k+1}$,

$$\begin{aligned}\tau_\delta w_{n-\delta-k} = (c_k + \lambda_k)w_1 + \ldots . (c_\delta + \lambda_\delta)w_{\delta-k+1}&\\ + (d_1\tau_1 + \ldots . d_{\delta-1}\tau_{\delta-1})w_{n-\delta-k+1}&.\end{aligned}$$

But, by the multiplicative properties of $\tau_1 \ldots . \tau_{\delta-1}$, the highest w occurring from the parenthesis is $w_{\delta-k+1}$, and the highest τ is $\tau_{\delta-k}$. Hence, $\tau_\delta w_{n-\delta-k}$ and, similarly, $w_{n-\delta-k}\tau_\delta$, conform to the given law.

Evidently in (8) and (8′) we can so choose $\lambda_1 \ldots . \lambda_\delta$ that the coefficients of w_i occurring in (8) are the negative of those in (8′). We saw that this could be done for Case I, $n \not> 2\delta + 1$, except for $w_{n-\delta-1}$, should it occur. In the present case it cannot occur, for since $n > 2\delta + 1$, $\delta < n - \delta - 1$.

It has, therefore, been proved that if a system of deficiency $(\delta-1)$ can be put in a form having certain multiplicative properties, one of deficiency δ can. But, by going through this demonstration with $\delta=1$, it will be seen that a system of deficiency 1 can so be put, as, indeed, has been shown by Scheffers. Hence, the theorem is true for $\delta=2, 3 \ldots.$

Consider $\tau_\alpha \tau_\beta$, where $\beta \ngtr \alpha$.

$$\begin{aligned}\tau_\alpha \tau_\beta &= (u_{z_\alpha} + a_2 w_2 + \ldots . a_{n-\delta-1} w_{n-\delta-1}) \tau_\beta \\ &= u_{z_\alpha} \tau_\beta + x\end{aligned}$$

where x can contain only

$$w_1 \ldots . w_\beta, \ \tau_1 \ldots . \tau_{\beta-1}.$$

$$\begin{aligned}u_{z_\alpha} \tau_\beta &= u_{z_\alpha} (u_{z_\beta} + b_2 w_2 + \ldots . b_{n-\delta-1} w_{n-\delta-1}) \\ &= u_{z_\alpha} (u_1, u_2 \ldots . u_{n-1}) = (u_1 \ldots . u_{z_\alpha-1}) \\ &= (w_1 \ldots . w_{n-\delta-1}, \ u_{z_1} \ldots . u_{z_{\alpha-1}}).\end{aligned}$$

Hence,

$$\tau_\alpha \tau_\beta = e_1 w_1 + \ldots e_\alpha w_\alpha + e_{\alpha+1} w_{\alpha+1} + \ldots e_{n-\delta-1} w_{n-\delta-1} + b_1 \tau_1 + \ldots b_{\alpha-1} \tau_{\alpha-1},$$

or

$$\begin{aligned}\tau_\alpha \tau_\beta = e_1 \sigma^{n-\delta-1} + \ldots . e_\alpha \sigma^{n-\delta-\alpha} + e_{\alpha+1} \sigma^{n-\delta-\alpha-1} \\ + \ldots . e_{n-\delta-1} \sigma + b_1 \tau_1 + \ldots . b_{\alpha-1} \tau_{\alpha-1}.\end{aligned} \tag{9}$$

If $n-\delta \ngtr \alpha+1$, $n-\delta-1 \ngtr \alpha$, and $e_m \, (m>\alpha)$ does not occur in (9). If $n-\delta > \alpha+1$, $\sigma^{\alpha+1} \neq 0$, and multiplying (9) by σ^α, since

$$\tau_1 \sigma^\alpha = \tau_2 \sigma^\alpha = \ldots . \tau_{\alpha-1} \sigma^\alpha = 0,$$

we find

$$\tau_\alpha \tau_\beta \sigma^\alpha = e_{\alpha+1} \sigma^{n-\delta-1} + \ldots . e_{n-\delta-1} \sigma^{\alpha+1}. \tag{10}$$

If $\beta < \alpha$, $\tau_\beta \sigma^\alpha = 0$, and the preceding equation necessitates

$$e_{\alpha+1} = e_{\alpha+2} = \ldots . e_{n-\delta-1} = 0.$$

Similarly for $\tau_\beta \tau_\alpha$. If, however, $\beta=\alpha$, (10) becomes

$$\tau_\alpha^2 \sigma^\alpha = e_{\alpha+1} \sigma^{n-\delta-1} + \ldots + e_{n-\delta-1} \sigma^{\alpha+1}. \tag{11}$$

But $\qquad \tau_\alpha \sigma^\alpha = p w_1 = p \sigma^{n-\delta-1}.$

So $\qquad \tau_\alpha^2 \sigma^\alpha = p \tau_\alpha \sigma^{n-\delta-1}.$

And since $n-\delta > \alpha+1$, $n-\delta-1 > \alpha$, so $\tau_\alpha \sigma^{n-\delta-1} = 0$. Therefore, $\tau_\alpha^2 \sigma^\alpha = 0$, and comparing with (11),

$$e_{\alpha+1} = e_{\alpha+2} = \dots . \, e_{n-\delta-1} = 0$$

as before.

Therefore, the following facts have been proved: *Every irreducible non-quaternion system without skew units can have selected as its units quantities* $w_1 \dots . \, w_{n-\delta-1}$, $\tau_1 \dots . \, \tau_\delta$, η *which have the multiplicative properties given below:*

$w_i w_k = w_{i+k-n+\delta}$ *or* 0 *according as* $i+k-n+\delta > 0$ *or* $i+k-n+\delta \not> 0$. $\tau_\alpha w_{n-\delta-k}$ *and* $w_{n-\delta-k}\tau_\alpha$ *are zero if* $k > \alpha$, *contain only* w_1 *if* $k = \alpha$, *while if* $k < \alpha$ *they are linear in* $w_1 \dots . \, w_{\alpha-k+1}$, $\tau_1 \dots . \, \tau_{\alpha-k}$. *The coefficient of the term in* w_i *occurring in* $\tau_\alpha w_{n-\delta-1}$ *is the negative of that in* $w_{n-\delta-1}\tau_\alpha$ *except for* $i = n-\delta-1$. $\tau_\alpha \tau_\beta$ *and* $\tau_\beta \tau_\alpha$ $(\beta \not> \alpha)$ *are linear functions of* $w_1 \dots . \, w_\alpha$, $\tau_1 \dots . \, \tau_{\alpha-1}$.

It will be seen, therefore, that the multiplication of the τ's with each other is not regular, but the remaining multiplicative properties are much simplified from Scheffers' general regular form. If δ should be large compared with $(n-\delta-1)$ the w-τ form will not be so simple as the w-u_s form, which is regular, but if δ is small compared with $(n-\delta-1)$ it will be much simpler. By application of the associative law it is easily seen that for $\delta = 2$ the w-τ form is regular if we place the unit τ_1 before $w_{n-\delta-1}$.

If the row and column τ_δ be deleted we have a table in $w_1 \dots . \, w_{n-\delta-1}$, $\tau_1 \dots . \, \tau_{\delta-1}$, η or $(n-1)$ units, of deficiency $(\delta-1)$, which nowhere contains τ_δ. Hence, *if we take every independent system of the type considered in* $(n-1)$ *units of deficiency* $(\delta-1)$ *in the* w-τ *form, and border it with a row and column* τ_δ *having the multiplicative properties given above, we shall obtain every possible system in n units of deficiency* δ.

This bordering will introduce certain parameters, which can be reduced in number by application of the associative law and also by the fact that $\tau_\delta^{n-\delta} = 0$. Further reductions may be made by introducing new units of such a type as not to change the multiplicative properties. Allowable substitutions are

$$\begin{aligned} \tau'_\alpha &= a_\alpha \tau_\alpha + b_\alpha \tau_{\alpha-1} + \dots . \, p_\alpha \tau_1, \\ \sigma' &= a\sigma \; + b\sigma^2 \; + \dots . \, p\sigma^{n-\delta-1}. \end{aligned}$$

Other substitutions may be available in special cases. If $n > 3\delta$, applications of the associative law can be made irrespective of n. If, however, $n \not> 3\delta$, each value of n has to be considered separately.

It will be noticed that with a given original set of units and with σ once chosen the τ's were determined uniquely. The new τ's indicated by the first of the preceding equations are those which could arise from the same choice of σ, but with the given system in a different form, linearly connected with the old form.

It may be asked, conversely, are all systems in the w-τ form non-quaternion systems of the type considered? If they are non-quaternion, they are of that type, from the form of the characteristic equation. It is easy to show from the multiplicative properties of the w-τ form that it is impossible to find three quantities, e_1, e_2, e_3, satisfying equations (1). Hence, no quaternion system can be put in that form.

§4.

A quantity ν such that

$$\nu x = x\nu = 0$$

for all values of x may be called a nilfactor. Such cannot exists in a system containing a modulus. The following theorem will be needed in §6, by incomplete systems being meant systems without a modulus: *If two incomplete systems are equivalent, and the same nilfactor is a unit in each, then if this nilfactor is deleted from each system the deleted systems are equivalent.*

Let the systems be

$$u_1\, u_2 \ldots . \, u_{n-1}\, \nu,$$
$$u'_1\, u'_2 \ldots . \, u'_{n-1}\, \nu',$$

where

$$\left.\begin{aligned} u'_i &= a_{i,\,1}\, u_1 + \ldots . \, a_{i,\,n-1}\, u_{n-1} + b_i\, \nu, \quad i = 1, 2 \ldots . \, n-1 \\ \nu' &= \nu, \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} u_i &= a'_{i,\,1}\, u'_1 + \ldots . \, a'_{i,\,n-1}\, u'_{n-1} + b'_i\, \nu', \quad i = 1, 2 \ldots . \, n-1 \\ \nu &= \nu'. \end{aligned}\right\} \tag{2}$$

Then the systems obtained by deleting ν and ν' are equivalent. For, consider the system

$$v'_i = a_{i,\,1}\, v_1 + \ldots . \, a_{i,\,n-1}\, v_{n-1}, \qquad i = 1, 2 \ldots . \, n-1 \tag{3}$$

the system $v_1 \dots\dots v_{n-1}$ having the multiplication table of $u_1 \dots\dots u_{n-1}$ with ν deleted. Now, equations (2) can be obtained from equations (1) by substituting ν' for ν in the first $(n-1)$ equations of (1) and then eliminating for the quantities $u_1 \dots\dots u_{n-1}$. Evidently the coefficient of u'_j occurring in u_i of (2) will be unaffected by the values of $b_1 \dots\dots b_{n-1}$ in (1), hence will be the same if $b_1 \dots\dots b_{n-1}$ are zero. Therefore, we obtain from (3)

$$v_i = a'_{i,1} v'_1 + \dots\dots a'_{i,n-1} v'_{n-1}, \qquad i = 1, 2 \dots\dots n-1. \quad (4)$$

ν being a nilfactor,

$$u'_i u'_j = (a_{i,1} u_1 + \dots\dots a_{i,n-1} u_{n-1})(a_{j,1} u_1 + \dots\dots a_{j,n-1} u_{n-1}).$$

Now the multiplicative properties of the u's after deletion are the same as before, except that ν is dropped out. Hence, $v'_i v'_j$ expressed in v's is identical with $u'_i u'_j$ expressed in u's (with $v_k = u_k$), except for the terms in ν. Therefore, from (2) and (4), $v'_i v'_j$ expressed in v''s is identical with $u'_i u'_j$ expressed in u''s (if we set $v'_k = u'_k$), except for terms in ν'. Hence, the multiplication table of the v''s is identical with that of the u''s with ν' deleted, or the deleted u' table can be obtained from the deleted u table by

$$u'_i = a_{i,1} u_1 + \dots\dots a_{i,n-1} u_{n-1}, \qquad i = 1, 2 \dots\dots n-1$$

which proves the theorem.

This proof evidently holds if, instead of having $\nu' = \nu$, we have $\nu' = c\nu$. The theorem is also true if $\nu' = c\nu + x$, $(c \neq 0)$, provided one of ν and ν', say ν, occurs nowhere in the products $u_i u_j$. For if we substitute for ν a new unit $\nu'' = c\nu + x$, then $\nu' = \nu''$, and ν' and ν'' can be deleted. But since ν does not occur in $u_i u_j$, no change is made in the table, and hence the deletion of ν'' has the same effect as that of ν.

§5.

An application of the principles obtained will now be made to the case $\delta = 2$. $n > 2$, else the characteristic equation is of degree lower than unity, which is impossible. If $n = 3$, the characteristic equation becomes $x - \xi\eta = 0$ or $x = \xi\eta$, or there is only one unit in the system, η, not three. The cases $n = 4$, $n = 5$ have been already considered by Scheffers by different methods. These will, therefore, not be considered here except for comparison with certain results true for $n > 5$.

n being greater than five, the general multiplicative properties of the $w - \tau$ form give us the following table, η being omitted as unnecessary:

	w_1	$w_2 \ldots\ldots w_{n-5}$	w_{n-4}	w_{n-3}	τ_1	τ_2
w_1	0	$0 \ldots\ldots 0$	0	0	0	0
w_2	0	$0 \ldots\ldots 0$	0	w_1	0	0
$\vdots$	$\vdots$	$\vdots \qquad \vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w_{n-5}	0	$0 \ldots\ldots w_{n-8}$	w_{n-7}	w_{n-6}	0	0
w_{n-4}	0	$0 \ldots\ldots w_{n-7}$	w_{n-6}	w_{n-5}	0	lw_1
w_{n-3}	0	$w_1 \ldots\ldots w_{n-6}$	w_{n-5}	w_{n-4}	aw_1	$c\tau_1 + ew_1 + kw_2$
τ_1	0	$0 \ldots\ldots 0$	0	$-aw_1$	bw_1	$r\tau_1 + fw_1 + pw_2$
τ_2	0	$0 \ldots\ldots 0$	mw_1	$d\tau_1$ $- ew_1$ $- kw_2$	$s\tau_1$ $+ gw_1$ $+ qw_2$	$j\tau_1 + hw_1 + iw_2$

By comparison of $(\sigma\tau_2)\sigma$ with $\sigma(\tau_2\sigma)$, it follows that $k = \dfrac{a(c+d)}{2}$, whence

$$l = \frac{a}{2}(3c+d), \quad m = -\frac{a}{2}(3d+c).$$

By comparing $\tau_2^2\tau_1$ with $\tau_2(\tau_2\tau_1)$ and $\tau_1\tau_2^2$ with $(\tau_1\tau_2)\tau_2$, there results $r = s = 0$. It is therefore evident that *the table will be regular in Scheffers' sense by placing* τ_1 *before* w_{n-3}; *also* w_1 *is a nilfactor.* These facts are true for $n = 5$, and, if σ is chosen suitably, for $n = 4$.

Further application of the associative law must be made separately for $n = 6$ and $n > 6$. An additional property common to both cases is found to be $p = q = 0$. This gives rise to the important fact that w_1 is unique; that is, *if two systems*

$$w_1 \ldots\ldots w_{n-3},\ \tau_1,\ \tau_2,\ \eta, \text{ and } w'_1 \ldots\ldots w'_{n-3},\ \tau'_1,\ \tau'_2,\ \eta$$

are equivalent, then w'_1 *equals* w_1 *except for a constant factor.* For

$$\sigma' = A_1\sigma + \ldots\ldots A_{n-3}\sigma^{n-3} + B_1\tau_1 + B_2\tau_2 + C\eta.$$

C must be zero, else $\sigma'^{n-2} \neq 0$. Therefore, σ'^2 can contain only $\sigma^2 \dots \sigma^{n-3}$, τ_1. For the products of the σ's can contain only $\sigma^2 \dots\dots \sigma^{n-3}$, while the products of the τ's with each other and with the σ's can only contain w_1, w_2, τ_1; or σ^{n-3}, σ^{n-4}, τ_1; since $n > 5$, $n - 4 > 1$. Also $n - 3 > 2$, hence neither σ' nor σ'^2 can be w_1'.

Similarly, σ'^3 can contain only $\sigma^3 \dots\dots \sigma^{n-3}$, for $\tau_1\sigma^k = \sigma^k\tau_1 = 0$, $k > 1$, $\tau_2\sigma^k = \sigma^k\tau_2 = 0$, $k > 2$, while $\tau_2\sigma^2$, $\sigma^2\tau_2$, $\tau_2\tau_1$ and $\tau_1\tau_2$ contain only w_1, or σ^{n-3}, and $n - 3 > 2$. σ'^4 is accordingly linear in $\sigma^4 \dots\dots \sigma^{n-3}$, σ'^5 is linear in $\sigma^5 \dots\dots \sigma^{n-3}$, and so on up to σ'^{n-3}, which will contain only σ^{n-3}. That is, w_1' is the same as w_1, except for a constant factor. This theorem is not true for $n < 6$.

The system is now in the following form, only those products being given which are not definitely known.

	w_{n-4}	w_{n-3}	τ_1	τ_2
w_{n-4}			0	$\frac{a}{2}(3c + d)\, w_1$
w_{n-3}			aw_1	$c\tau_1 + \frac{a}{2}(c + d)\, w_2 + ew_1$
τ_1	0	$-aw_1$	bw_1	fw_1
τ_2	$-\frac{a}{2}(3d + c)\, w_1$	$d\tau_1 - \frac{a}{2}(c+d)\, w_2 - ew_1$	gw_1	$j\tau_1 + iw_2 + hw_1$

The remaining facts yielded by the associative law and the characteristic equation are mostly conditional, and subdivide the tables into classes. If $n > 6$ we have the following forms, reciprocal systems being considered equivalent:

I. If $b \neq 0$. Then $c = d = i = j = 0$.
II. If $b = 0, f^2 \neq g^2$. Then $c = d = i = j = 0$.
III. If $b = 0, f = g = a = 0$. Then $i = 0$.
IV. If $b = 0, f = g = 0, a \neq 0$. Then $i = j = 0$.
V. If $b = 0, f = g \neq 0, a = 0$. Then $c = d$, $i = cf$.
VI. If $b = 0, f = g \neq 0, a \neq 0$. Then $c = d, j = 0, i = cf$.
VII. If $b = 0, f = -g \neq 0$. Then $d = -c$, $i = cf, j = 0$.

By the transformation $\tau_2' = \tau_2 + a\tau_1$, h can be made zero in I, II, V and VI, and e can be made zero in IV. With these simplifications the preceding forms will be called typical forms.

The case $n = 6$ presents some especial difficulties, and its details will not be considered in this paper. For the writer's present purposes it will suffice to state that if w_2 does not enter into τ_2^2, $\tau_2\sigma$ or $\sigma\tau_2$, the system can be put into one of the following typical forms:

I. $b \neq 0;\ j = c = d = h = 0.$
II. $b = 0,\ a \neq 0,\ c = 0;\ d = e = j = 0.$
III. $b = 0,\ a \neq 0,\ c \neq 0;\ d = -c,\ j = f = g = e = 0.$
IV. $b = 0,\ a = 0,\ c \neq 0;\ f = g = 0.$
V. $b = 0,\ a = 0,\ c = d = 0,\ f \neq g;\ j = 0.$
VI. $b = 0,\ a = 0,\ c = d = 0,\ f = g.$

The cases $n = 5$ and $n = 4$ are very simple, and the following may be put as typical forms:

$n = 5$.

	w_1	w_2	τ_1	τ_2
w_1	0	0	0	0
w_2	0	w_1	aw_1	$c\tau_1 + ew_1$
τ_1	0	$-aw_1$	bw_1	fw_1
τ_2	0	$d\tau_1$ $-ew_1$	gw_1	$j\tau_1 + hw_1$

I. $b \neq 0;\ c = d = j = h = 0.$
II. $b = 0,\ a \neq 0;\ c = d = e = j = 0.$
III. $b = a = 0,\ f \neq 0;\ c = d = j = 0.$
IV. $b = a = f = 0;\ g = 0.$

For $n = 4$, in the preceding table w_2 does not exist, and $b = j = h = 0$, $g = -f$.

A quantity α such that $\alpha x = -x\alpha$ for all values of x is called an alternate. Such cannot exist in a system with a modulus. A nilfactor is thus also an alternate. By actual trial in each of the given typical forms the following theorems can be demonstrated:

I. *A system in a typical form possesses no nilfactors except linear functions of* w_1 *and such* τ*'s as are themselves nilfactors.*

II. *A system in a typical form possesses no alternates except linear functions of* w_1 *and such* τ*'s as are themselves alternates.*

§6.

We will next proceed to reduce the parameters so far as possible for $n > 6$. By means of the transformation $\tau_1' = \tau_1 + a\tau_2$, or by an interchange of the τ's, or both, Case I can be reduced to Case II if $f^2 \neq g^2$, to Cases V or VI (according as $e = 0$ or $e \neq 0$) if $f = g \neq 0$, to Case VII if $f = -g \neq 0$, and to Cases III or IV (according as $e = 0$ or $e \neq 0$) if $f = g = 0$.

Transformations $\tau_2' = x\tau_2$, $\tau_1' = y\tau_1$, $\sigma' = z\sigma$ enable us to reduce a great many parameters to unity or zero, dividing each typical form into a large number of subcases. Certain transformations show some of these subcases to be equivalent to others of the same group. Thus in III are employed

$$\sigma' = \sigma + a\tau_2 \text{ and } \tau_1' = \tau_1 + aw_1,$$

and in VII,

$$\tau_2' = \tau_2 + a\tau_1,\ \sigma' = \sigma + a\sigma^2, \text{ and } \sigma' = \sigma + a\tau_1.$$

Furthermore, an interchange of the τ's shows an equivalence between certain cases of different groups.

There is thus found a total of 40 systems. To test directly to see what of these are linearly independent might require 780 applications of the general linear transformation. As a matter of fact it would require at least 364. The process is greatly reduced by the following considerations:

Suppose two systems $w_1 \ldots w_{n-3}, \tau_1, \tau_2, \eta$, and $w_1' \ldots w_{n-3}', \tau_1', \tau_2'; \eta'$ are equivalent. Evidently $\eta = \eta'$. Consider any unit u' of the second system different from η'.

$$u' = a_1 w_1 + \ldots . a_{n-3} w_{n-3} + b_1 \tau_1 + b_2 \tau_2 + c\eta.$$

But c must be zero, else $u'^{n-2} \neq 0$. Hence, the incomplete systems $w_1 \ldots . w_{n-3}, \tau_1, \tau_2$ and $w_1' \ldots . w_{n-3}', \tau_1', \tau_2'$ are equivalent. We therefore need test only the incomplete systems.

First, the systems can be divided according as they are commutative or non-commutative. Secondly, since the number of linearly independent nilfactors is evidently a characteristic of the incomplete system, by the theorem on p. 382 these two groups can be divided according as none, one, or two of the τ's are nilfactors, the last case of which can occur, of course, only in the commutative class. Thirdly, since the number of linearly independent alternates is evidently a characteristic of the incomplete system, the subgroups of the non-commutative class can be subdivided according as none, one, or two of the τ's are alternates. These considerations separate the systems into eight distinct classes.

Next, supposing two systems to be equivalent, delete w_1. w_1 being unique and a nilfactor, the deleted systems must be equivalent by the theorem on page 377. The deleted systems will be in the typical w-τ forms with w_2 taking the place of w_1, and will be of deficiency 2 with one less unit. We can, therefore, subdivide each class according to the commutative, nilfactive, or alternate properties of the τ's with w_1 deleted.

Now, if two systems, with w_1 deleted, are equivalent, w_2 is unique, for it takes the place of w_1, and, since $n > 6$, the deleted system has at least six units (counting η). This does not imply that w_2 was originally equal to w_2', for they may have differed by the deleted w_1. Therefore, w_2 being a nilfactor, if we delete it, the deleted systems must be equivalent. The deleted systems will be in the typical w-τ forms with w_3 taking the place of w_1, and will still be of deficiency 2. We can, therefore, make subdivisions according to the commutative, nilfactive, or alternate properties of the τ's with w_1 and w_2 deleted. In some of the classes w_2 does not enter into any products, and in fact the application of this principle gives only three separations.

We can proceed thus to delete w's until we have only the four units w_{n-4}, w_{n-3}, τ_1, τ_2, although the deletions after w_2 change no multiplicative property of the τ's. w_{n-4} is not necessarily unique in the deleted systems (see theorem, p. 379), so it cannot be deleted. Now, if originally two systems were equivalent, they will be equivalent thus deleted. We can, consequently, apply the general linear transformation in four units to test the equivalence of the systems. The results, in common with those given in the last few paragraphs, are only negative; that is, equivalence in the deleted systems does not imply equivalence in the original systems, although non-equivalence necessitates non-equivalence originally.

Application of the general linear transformation in w_{n-4}, w_{n-3}, τ_1, τ_2 shows that a system in which $j = 0$ is distinct from one in which $j \neq 0$, and a system in which $j = c = 0$ or $j = d = 0$ is distinct from one in which d, $c \neq 0$. These lead to further classifications and divide the systems into twenty-seven distinct groups, necessitating at most fifty applications of the general linear transformation to test equivalence.

The general linear transformation is

$$\begin{aligned} w'_{n-3} \equiv \sigma' &= a_1 \sigma + a_2 \sigma^2 + \ldots a_{n-3} \sigma^{n-3} + b_1 \tau_1 + b_2 \tau_2, \\ \tau'_1 &= A_1 \sigma + A_2 \sigma^2 + \ldots A_{n-3} \sigma^{n-3} + B_1 \tau_1 + B_2 \tau_2, \\ \tau'_2 &= \alpha_1 \sigma + \alpha_2 \sigma^2 + \ldots \alpha_{n-3} \sigma^{n-3} + \beta_1 \tau_1 + \beta_2 \tau_2. \end{aligned}$$

$a_1 \neq 0$, else $\sigma'^{n-3} = 0$. $w'_{n-4} \ldots w'_1$ follow as powers of σ'. From the general multiplicative properties of the system w'_{n-4} can contain only $\sigma^2 \ldots \sigma^{n-3}$, τ_1, and $w'_{n-(4+k)}$ only $\sigma^{2+k} \ldots \sigma^{n-3}$, and in each case the first term must occur, its coefficient being a_1^{2+k}. In fact

$$w'_1 = a_1^{n-3} w_1 \qquad w'_2 = a_1^{n-4} w_2 + b w_1 \qquad w'_3 = a_1^{n-5} w_3 + c w_2 + d w_1 + e \tau_1$$

where $e = 0$ unless $n = 7$, and b, c, d and e are independent of $a_4 \ldots a_{n-3}$.

The products of τ'_1, τ'_2 with $\sigma'^{n-4} \ldots \sigma'^2$ show successively that $A_1, A_2 \ldots A_{n-6}$, $\alpha_1, \alpha_2 \ldots \alpha_{n-6}$ are all zero, and these conditions suffice to make the whole w'-τ' table of proper form except the products of τ'_1 and τ'_2 with themselves, each other, and with σ' and σ'^2. Changing the notation, we have accordingly

$$\begin{aligned} \sigma' &= a_1 \sigma + a_2 \sigma^2 + \ldots a_{n-3} \sigma^{n-3} + b_1 \tau_1 + b_2 \tau_2, \\ \tau'_1 &= A_1 w_1 + A_2 w_2 + A_3 w_3 + B_1 \tau_1 + B_2 \tau_2, \\ \tau'_2 &= \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 + \beta_1 \tau_1 + \beta_2 \tau_2. \end{aligned}$$

Now, the products of τ'_1 and τ'_2 with σ', $\sigma'^2 \ldots \sigma'^{n-3}$ expressed in w's and τ's, are not affected by the values of $a_4 \ldots a_{n-3}$. It follows from the two preceding sets of equations that the values of w_1, w_2, w_3, τ_1 and τ_2 expressed in w'_1, w'_2, w'_3, τ'_1, τ'_2 are independent of $a_4 \ldots a_{n-3}$. Therefore the multiplication table for $w'_1 \ldots w'_{n-3}$, τ'_1, τ'_2 is unaffected by making $a_4 = a_5 = \ldots a_{n-3} = 0$. Thus the transformations are simplified.

By these transformations the writer has tested the various forms for equivalence and non-equivalence, and has obtained the following table which comprises all the possible non-equivalent forms, provided any system is always regarded as the equivalent of its reciprocal, the letters having the signification given in the form on p. 380:

a	b	c	d	e	f	g	h	i	j
0	0	0	0	0	1	1	0	0	1
0	0	1	1	0	1	1	0	1	0
0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	g	0	0	0
1	0	0	0	1	1	1	0	0	0
1	0	0	0	0	1	1	0	0	0
0	0	0	0	0	1	-1	1	0	0
0	0	0	0	1	1	1	0	0	1
0	0	1	1	1	1	1	0	1	0
1	0	1	1	e	1	1	0	1	0
1	0	1	-1	0	1	-1	1	1	0
0	0	1	-1	0	1	-1	0	1	0
1	0	c	$2-c$	0	0	0	h	0	0
1	0	1	-1	0	1	-1	0	1	0
1	0	1	-1	0	0	0	h	0	0
1	0	0	0	0	0	0	1	0	0
0	0	1	d	0	0	0	1	0	0
0	0	1	d	0	0	0	0	0	0
0	0	1	0	0	0	0	1	0	1
0	0	0	1	0	0	0	0	0	1
0	0	0	0	1	0	0	0	0	1
0	0	1	1	1	0	0	0	0	0
0	0	1	1	1	0	0	1	0	0
0	0	0	0	1	0	0	1	0	0
1	0	0	0	0	0	0	0	0	0

To these must be added, if we do not regard reciprocals as necessarily equivalent:

a	b	c	d	e	f	g	h	i	j
0	0	0	1	0	0	0	1	0	0
0	0	0	1	0	0	0	0	0	0

In one of the systems e enters as a parameter. It is a little remarkable that it can be made zero except for the particular case $n = 8$, the necessary transformations containing a fraction with $(n - 8)$ in the denominator. Similarly, in the two systems in which h enters as a parameter, it can be reduced to unity if it is not zero, except for the particular case $n = 7$.

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